Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE EXAMINATION, 2004

MATHEMATICS — HIGHER LEVEL

PAPER 1 (300 marks)

THURSDAY, 10 JUNE – MORNING, 9:30 to 12:00

Attempt **SIX QUESTIONS** (50 marks each).

WARNING: Marks will be lost if all necessary work is not clearly shown.

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1. (a) Express $\frac{1-\sqrt{3}}{1+\sqrt{3}}$ in the form $a\sqrt{3}-b$, where a and $b \in \mathbf{N}$.

(b) (i) Let $f(x) = x^3 + kx^2 - 4x - 12$, where k is a constant.

Given that $x+3$ is a factor of $f(x)$, find the value of k.

(ii) Show that

$$\frac{3}{1+x^p} + \frac{3}{1+x^{-p}} \quad \text{simplifies to a constant.}$$

(c) (i) Show that $p^3 + q^3 - (p+q)^3 = -3pq(p+q)$.

(ii) Hence, or otherwise, find, in terms of a and b, the three values of x for which $(a-x)^3 + (b-x)^3 - (a+b-2x)^3 = 0$.

2. (a) Solve, without using a calculator, the following simultaneous equations:

$$3x + y + z = 0$$
$$x - y + z = 2$$
$$2x - 3y - z = 9.$$

(b) (i) Solve the inequality $\frac{x+1}{x-1} < 4$, where $x \in \mathbf{R}$ and $x \neq 1$.

(ii) The roots of $x^2 + px + q = 0$ are α and β, where $p, q \in \mathbf{R}$.

Find the quadratic equation whose roots are $\alpha^2\beta$ and $\alpha\beta^2$.

(c) (i) $f(x) = 2x + 1$, for $x \in \mathbf{R}$.
Show that there exists a real number k such that for all x,
$f(x + f(x)) = kf(x)$.

(ii) Show that for any real values of a, b and h, the quadratic equation

$$(x-a)(x-b) - h^2 = 0$$

has real roots.

3. **(a)** Find the real numbers p and q such that

$$2(p+iq)+i(p-iq)=5+i\text{, where } i^2=-1.$$

(b) **(i)** $z_1=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$ and $z_2=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$.

Evaluate z_1z_2, giving your answer in the form $x+iy$.

(ii) $w_1=a+ib$ and $w_2=c+id$.

Prove that $\overline{(w_1w_2)}=\left(\overline{w_1}\right)\left(\overline{w_2}\right)$,

where $\overline{w}$ is the complex conjugate of w.

(c) Let $A=\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}$ and $P=\begin{pmatrix}4 & 3\\ -2 & -1\end{pmatrix}$.

(i) Evaluate $A^{-1}PA$ and hence $\left(A^{-1}PA\right)^{10}$.

(ii) Use the fact that $\left(A^{-1}PA\right)^{10}=A^{-1}P^{10}A$ to evaluate P^{10}.

4. **(a)** Show that $3\binom{n}{3}=n\binom{n-1}{2}$ for all natural numbers $n\geq 3$.

(b) **(i)** Show that $\dfrac{2}{(2r-1)(2r+1)}=\dfrac{1}{2r-1}-\dfrac{1}{2r+1}$, $r\neq\pm\frac{1}{2}$.

(ii) Hence, find $\displaystyle\sum_{r=1}^{n}\frac{2}{(2r-1)(2r+1)}$.

(iii) Evaluate $\displaystyle\sum_{r=1}^{\infty}\frac{2}{(2r-1)(2r+1)}$.

(c) **(i)** The sequence $u_1,u_2,u_3,\ldots\ldots$ is given by $u_{n+1}=\sqrt{4-(u_n)^2}$ and $u_1=a>0$.

For what value of a will all of the terms of the sequence be equal to each other?

(ii) p,q and r are three numbers in arithmetic sequence.

Prove that $p^2+r^2\geq 2q^2$.

5. **(a)** Find the fifth term in the expansion of

$$\left(x^2 - \frac{1}{x}\right)^6$$

and show that it is independent of x.

(b) **(i)** In a geometric series, the second term is 8 and the fifth term is 27. Find the first term and the common ratio.

(ii) Solve $\log_4 x - \log_4 (x-2) = \frac{1}{2}$.

(c) Prove by induction that $2^n \geq n^2$, $n \in \mathbf{N}$, $n \geq 4$.

6. **(a)** Differentiate $\frac{1}{2+5x}$ with respect to x.

(b) **(i)** Given $y = \tan^{-1} x$, find the value of $\frac{dy}{dx}$ at $x = \sqrt{2}$.

(ii) Differentiate, from first principles, $\cos x$ with respect to x.

(c) Let $f(x) = x^3 + 6x^2 + 15x + 36$, $x \in \mathbf{R}$.

(i) Show that $f'(x)$ can be written in the form $3[(x+a)^2 + b]$, $a, b \in \mathbf{R}$, where $f'(x)$ is the first derivative of $f(x)$.

(ii) Hence show that $f(x) = 0$ has only one real root.

7. **(a)** An object's distance from a fixed point is given by $s = 12 + 24t - 3t^2$, where s is in metres and t is in seconds.
Find the speed of the object when $t = 3$ seconds.

(b) The parametric equations of a curve are:

$$x = 2\theta - \sin 2\theta$$
$$y = 1 - \cos 2\theta \quad , \text{ where } 0 < \theta < \pi.$$

(i) Find $\dfrac{dy}{dx}$.

(ii) Show that the tangent to the curve at $\theta = \dfrac{\pi}{6}$ is perpendicular to the tangent at $\theta = \dfrac{2\pi}{3}$.

(c) Given that $x = \dfrac{e^{2y} - 1}{e^{2y} + 1}$,

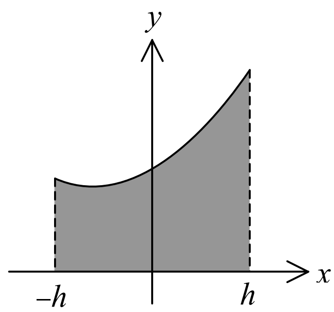
(i) show that $e^{2y} = \dfrac{1+x}{1-x}$

(ii) show that $\dfrac{dy}{dx}$ can be expressed in the form $\dfrac{p}{1-x^q}$, $p, q \in \mathbf{N}$.

8. **(a)** Find **(i)** $\displaystyle\int \frac{1}{x^2}dx$ **(ii)** $\displaystyle\int \cos 6x dx$.

(b) Evaluate **(i)** $\displaystyle\int_3^6 \frac{dx}{\sqrt{36 - x^2}}$ **(ii)** $\displaystyle\int_0^{\frac{\pi}{3}} \sin x \cos^3 x dx$.

(c) The graph of the function $f(x) = ax^2 + bx + c$ from $x = -h$ to $x = h$ is shown in the diagram.



(i) Show that the area of the shaded region is
$$\frac{h}{3}\left[2ah^2 + 6c\right].$$

(ii) Given that $f(-h) = y_1$, $f(0) = y_2$ and $f(h) = y_3$, express the area of the shaded region in terms of y_1, y_2, y_3 and h.

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