Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE EXAMINATION, 2005

MATHEMATICS – HIGHER LEVEL

PAPER 1 (300 marks)

THURSDAY, 9 JUNE – MORNING, 9:30 to 12:00

Attempt **SIX QUESTIONS** (50 marks each).

WARNING: **Marks will be lost if all necessary work is not clearly shown.**

Answers should include the appropriate units of measurement, where relevant.

1. **(a)** Solve the simultaneous equations:

$$\frac{x}{5}-\frac{y}{4}=0$$

$$3x+\frac{y}{2}=17.$$

(b) **(i)** Express $2^{\frac{1}{4}}+2^{\frac{1}{4}}+2^{\frac{1}{4}}+2^{\frac{1}{4}}$ in the form $2^{\frac{p}{q}}$, where $p, q \in \mathbf{Z}$.

(ii) Let $f(x)=ax^3+bx^2+cx+d$.
Show that $(x-t)$ is a factor of $f(x)-f(t)$.

(c) $(x-p)^2$ is a factor of x^3+qx+r.
Show that $27r^2+4q^3=0$.
Express the roots of $3x^2+q=0$ in terms of p.

2. **(a)** Solve for x: $\left|x-1\right|<7$, where $x \in \mathbf{R}$.

(b) The cubic equation $4x^3+10x^2-7x-3=0$ has one integer root and two irrational roots. Express the irrational roots in simplest surd form.

(c) Let $f(x)=\dfrac{x^2+k^2}{mx}$, where k and m are constants and $m \neq 0$.

(i) Show that $f(km)=f\left(\dfrac{k}{m}\right)$.

(ii) a and b are real numbers such that $a \neq 0, b \neq 0$ and $a \neq b$.
Show that if $f(a)=f(b)$, then $ab=k^2$.

3. **(a)** Given that $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, show that $A^3 = A^{-1}$.

(b) Solve the quadratic equation:

$$2iz^2 + (6+2i)z + (3-6i) = 0, \text{ where } i^2 = -1.$$

(c) **(i)** $z = \cos\theta + i\sin\theta$. Use De Moivre's theorem to show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \text{ for } n \in \mathbf{N}.$$

(ii) Expand $\left(z + \frac{1}{z}\right)^4$ and hence express $\cos^4\theta$ in terms of $\cos 4\theta$ and $\cos 2\theta$.

4. **(a)** Write the recurring decimal 0.636363….. as an infinite geometric series and hence as a fraction.

(b) **(i)** The first three terms in the binomial expansion of $(1+kx)^n$ are $1 - 21x + 189x^2$.
Find the value of n and the value of k.

(ii) A sequence is defined by $u_n = (2-n)2^{n-1}$.
Show that $u_{n+2} - 4u_{n+1} + 4u_n = 0$, for all $n \in \mathbf{N}$.

(c) **(i)** Show that

$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}, \text{ where } a \text{ and } b \text{ are real numbers.}$$

(ii) The lengths of the sides of a right-angled triangle are a, b and c, where c is the length of the hypotenuse.
Using the result from part **(i)**, or otherwise, show that $a + b \leq c\sqrt{2}$.

5. **(a)** Solve for x: $\sqrt{10-x} = 4 - x$.

(b) Prove by induction that

$$\sum_{r=1}^{n}(3r-2) = \frac{n}{2}(3n-1).$$

(c) **(i)** Show that $\dfrac{1}{\log_a b} = \log_b a$, where $a, b > 0$ and $a, b \neq 1$.

(ii) Show that

$$\frac{1}{\log_2 c} + \frac{1}{\log_3 c} + \frac{1}{\log_4 c} + \ .... + \frac{1}{\log_r c} = \frac{1}{\log_{r!} c}, \text{ where } c > 0, c \neq 1.$$

6. **(a)** Differentiate with respect to x

(i) $(1+7x)^3$ **(ii)** $\sin^{-1}\left(\dfrac{x}{5}\right)$.

(b) Let $y = \dfrac{1-\cos x}{1+\cos x}$.

Show that $\dfrac{dy}{dx} = t + t^3$, where $t = \tan\dfrac{x}{2}$.

(c) The equation of a curve is $y = \dfrac{x}{x-1}$, where $x \neq 1$.

(i) Show that the curve has no local maximum or local minimum point.

(ii) Write down the equations of the asymptotes and hence sketch the curve.

(iii) Show that the curve is its own image under the symmetry in the point of intersection of the asymptotes.

7. **(a)** Find from first principles the derivative of x^2 with respect to x.

(b) **(i)** The parametric equations of a curve are:

$$x = 8 + \ln t^2$$
$$y = \ln(2 + t^2), \text{ where } t > 0.$$

Find $\dfrac{dy}{dx}$ in terms of t and calculate its value at $t = \sqrt{2}$.

(ii) Find the slope of the tangent to the curve $xy^2 + y = 6$ at the point (1, 2).

(c) **(i)** Write down a quadratic equation whose roots are $\pm\sqrt{k}$.

(ii) Hence use the Newton-Raphson method to show that the rule

$$u_{n+1} = \frac{(u_n)^2 + k}{2u_n}$$

can be used to find increasingly accurate approximations for $\sqrt{k}$.

(iii) Using the above rule and taking $\dfrac{3}{2}$ as the first approximation for $\sqrt{3}$, find the third approximation, as a fraction.

8. **(a)** Find **(i)** $\int (2 + x^3)dx$ **(ii)** $\int e^{3x}\,dx.$

(b) **(i)** Evaluate $\displaystyle\int_1^4 \frac{2x+1}{x^2+x+1}\,dx$.

(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{8}} \sin^2 2\theta \, d\theta.$

(c) **(i)** Evaluate $\displaystyle\int_1^2 \frac{1}{\sqrt{3+2x-x^2}}\,dx$.

(ii) Use integration methods to derive a formula for the volume of a cone.

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