Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE EXAMINATION, 2010

MATHEMATICS – HIGHER LEVEL

PAPER 1 (300 marks)

FRIDAY, 11 June – AFTERNOON, 2:00 to 4:30

Attempt **SIX QUESTIONS** (50 marks each).

WARNING: **Marks will be lost if all necessary work is not clearly shown.**

Answers should include the appropriate units of measurement, where relevant.

1. **(a)** $x^2 - 6x + t = (x + k)^2$, where t and k are constants.
Find the value of k and the value of t.

(b) Given that p is a real number, prove that the equation $x^2 - 4px - x + 2p = 0$ has real roots.

(c) $(x-2)$ and $(x+1)$ are factors of $x^3 + bx^2 + cx + d$.

(i) Express c in terms of b.

(ii) Express d in terms of b.

(iii) Given that b, c and d are three consecutive terms in an arithmetic sequence, find their values.

2. **(a)** Solve the simultaneous equations

$$\begin{aligned} 2x + 3y &= 0 \\ x + y + z &= 0 \\ 3x + 2y - 4z &= 9. \end{aligned}$$

(b) The equation $x^2 - 12x + 16 = 0$ has roots α^2 and β^2, where $\alpha > 0$ and $\beta > 0$.

(i) Find the value of $\alpha\beta$.

(ii) Hence, find the value of $\alpha + \beta$.

(c) **(i)** Prove that for all real numbers a and b,

$$a^2 - ab + b^2 \geq ab.$$

(ii) Let a and b be non-zero real numbers such that $a + b \geq 0$.

Show that $\dfrac{a}{b^2} + \dfrac{b}{a^2} \geq \dfrac{1}{a} + \dfrac{1}{b}$.

3. **(a)** Find x and y such that

$$\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 20 \\ 32 \end{pmatrix}.$$

(b) Let $z_1 = s + 8i$ and $z_2 = t + 8i$, where $s \in \mathbb{R}, t \in \mathbb{R}$, and $i^2 = -1$.

(i) Given that $|z_1| = 10$, find the possible values of s.

(ii) Given that $\arg(z_2) = \frac{3\pi}{4}$, find the value of t.

(c) **(i)** Use De Moivre's theorem to find, in polar form, the five roots of the equation

$$z^5 = 1.$$

(ii) Choose one of the roots w, where $w \neq 1$. Prove that $w^2 + w^3$ is real.

4. **(a)** Write the recurring decimal 0·474747….. as an infinite geometric series and hence as a fraction.

(b) In an arithmetic sequence, the fifth term is –18 and the tenth term is 12.

(i) Find the first term and the common difference.

(ii) Find the sum of the first fifteen terms of the sequence.

(c) **(i)** Show that $(r+1)^3 - (r-1)^3 = 6r^2 + 2$.

(ii) Hence, or otherwise, prove that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

(iii) Find $\sum_{r=11}^{30} (3r^2 + 1)$.

5. **(a)** Solve the equation: $\log_2(x+6)-\log_2(x+2)=1$.

(b) Use induction to prove that

$$2+(2\times 3)+(2\times 3^2)+(2\times 3^3)+\dots\dots\dots+(2\times 3^{n-1}) = 3^n-1,$$

where n is a positive integer.

(c) **(i)** Expand $\left(x+\frac{1}{x}\right)^2$ and $\left(x+\frac{1}{x}\right)^4$.

(ii) Hence, or otherwise, find the value of $x^4+\frac{1}{x^4}$, given that $x+\frac{1}{x}=3$.

6. **(a)** The equation $x^3+x^2-4=0$ has only one real root.
Taking $x_1=\frac{3}{2}$ as the first approximation to the root, use the Newton-Raphson method to find x_2, the second approximation.

(b) Parametric equations of a curve are:

$$x=\frac{2t-1}{t+2}, \qquad y=\frac{t}{t+2}, \qquad \text{where } t\in\mathbb{R}\setminus\{-2\}.$$

(i) Find $\frac{dy}{dx}$.

(ii) What does your answer to part **(i)** tell you about the shape of the graph?

(c) A curve is defined by the equation $x^2y^3+4x+2y=12$.

(i) Find $\frac{dy}{dx}$ in terms of x and y.

(ii) Show that the tangent to the curve at the point $(0,6)$ is also the tangent to it at the point $(3,0)$.

7. **(a)** Differentiate x^2 with respect to x from first principles.

(b) Let $y = \dfrac{\cos x + \sin x}{\cos x - \sin x}$.

(i) Find $\dfrac{dy}{dx}$.

(ii) Show that $\dfrac{dy}{dx} = 1 + y^2$.

(c) The function $f(x) = (1+x)\log_e(1+x)$ is defined for $x > -1$.

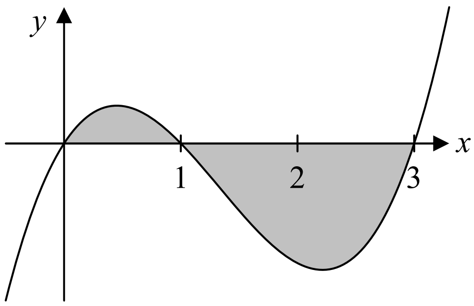
(i) Show that the curve $y = f(x)$ has a turning point at $\left(\dfrac{1-e}{e}, -\dfrac{1}{e}\right)$.

(ii) Determine whether the turning point is a local maximum or a local minimum.

8. **(a)** Find $\int \left(\sin 2x + e^{4x}\right)\, dx$.

(b) The curve $y = 12x^3 - 48x^2 + 36x$ crosses the x-axis at $x = 0$, $x = 1$ and $x = 3$, as shown.

Calculate the total area of the shaded regions enclosed by the curve and the x-axis.



(c) **(i)** Find, in terms of a and b,

$$I = \int_a^b \frac{\cos x}{1 + \sin x}\, dx$$

(ii) Find in terms of a and b,

$$J = \int_a^b \frac{\sin x}{1 + \cos x}\, dx.$$

(iii) Show that if $a + b = \dfrac{\pi}{2}$, then $I = J$.

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