Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE EXAMINATION, 2012

MATHEMATICS – HIGHER LEVEL

PAPER 1 (300 marks)

FRIDAY, 8 JUNE – AFTERNOON, 2:00 to 4:30

Attempt **SIX QUESTIONS** (50 marks each).

WARNING: **Marks will be lost if all necessary work is not clearly shown.**

Answers should include the appropriate units of measurement, where relevant.

1. **(a)** The following equation is true for all x.

$$ax^2 + bx(x-4) + c(x-4) = x^2 + 13x - 20.$$

Find the values of the constants a, b and c.

(b) The function $f(x) = x^3 - 2x^2 - 5x + 6$ has three integer roots.

(i) Find the three roots.

(ii) Find a cubic equation whose roots are 1 less than the roots of f.

(c) **(i)** Show that $kx - t$ is a factor of $k^3x^3 - k^2tx^2 + ktx - t^2$, where k and t are non-zero real constants.

(ii) Given any value of $k \neq 0$, find the set of values of t for which the equation $k^3x^3 - k^2tx^2 + ktx - t^2 = 0$ has three distinct real roots.

2. **(a)** Solve for x: $\sqrt{2x+3} = 2x - 3, \quad x \in \mathbb{R}$.

(b) α and β are the roots of the equation $x^2 - 2x + 5 = 0$.

(i) Find the value of $\alpha^2 + \beta^2$.

(ii) Find a quadratic equation whose roots are $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$.

(c) **(i)** Show that if x is a positive real number, then $x + \frac{1}{x} \geq 2$.

(ii) Show that if x is a negative real number, then $x + \frac{1}{x} \leq -2$.

(iii) Show that, for all $x \in \mathbb{R} \setminus \{0\}$, $\left| x^3 + \frac{1}{x^3} \right| \geq 2$.

3. **(a)** Verify that $z = 2 - 3i$ satisfies the equation $z^3 - z^2(2-3i) + z - 2 + 3i = 0$, where $i^2 = -1$.

(b) Let $A = \begin{pmatrix} 2y & y \\ x^2 & x \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$, where $x, y \in \mathbb{R}$.

(i) Find AB in terms of x and y.

(ii) Solve for x and y the equation $AB = \begin{pmatrix} -4 & 5 \\ 15 & -24 \end{pmatrix}$.

(c) z is a complex number such that $z^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

(i) Find the two possible values of z.

(ii) On an Argand diagram, the points representing $-z$, z and $z^2 + k$ are collinear, where $k \in \mathbb{R}$. Find the value of k.

4. **(a)** $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ are consecutive terms of an arithmetic sequence, where $a, b, c \in \mathbb{R} \setminus \{0\}$.

Express b in terms of a and c. Give your answer in its simplest form.

(b) **(i)** Show that $\frac{1}{\sqrt{r+1} + \sqrt{r}} = \sqrt{r+1} - \sqrt{r}$, for $r \geq 0$.

(ii) Find $\sum_{r=1}^{n} \frac{1}{\sqrt{r+1} + \sqrt{r}}$.

(iii) Evaluate $\sum_{r=1}^{99} \frac{1}{\sqrt{r+1} + \sqrt{r}}$.

(c) a, b and c are consecutive terms in a geometric sequence, where $a + b \neq 0$ and $b + c \neq 0$.

Show that $\frac{2ab}{a+b}$, b and $\frac{2bc}{b+c}$ are consecutive terms in an arithmetic sequence.

5. **(a)** Solve for $x \in \mathbb{R}$: $\log_4(2x+6) - \log_4(x-1) = 1$.

(b) Consider the binomial expansion of $\left(3x^2 + \frac{1}{2x}\right)^{10}$ in descending powers of x.

(i) Find an expression for the general term.

(ii) Find the coefficient of x^8.

(iii) Show that there is no term independent of x.

(c) **(i)** Prove that if $k \geq 4$, then $k^2 > 2k + 1$.

(ii) Prove by induction that, for all natural numbers $n \geq 4$, $2^n \geq n^2$.

6. **(a)** Differentiate with respect to x:

(i) $(4x^2 - 1)^3$.

(ii) $\sin^{-1}\left(\frac{2x}{3}\right)$.

(b) **(i)** Differentiate $\sqrt{x}$ with respect to x, from first principles.

(ii) Find the equation of the tangent to the curve $y = \sqrt{x}$ at the point $(9, 3)$.

(c) Let f be the function $f : x \to 8x + \sin 4x + 4\sin 2x$, where $x \in \mathbb{R}$.

(i) Find $f'(x)$.

(ii) Express $f'(x)$ in terms of $\cos 2x$.

(iii) Prove that $f(x)$ is increasing for all values of x.

7. **(a)** Given that $x = 3t^2 - 6t$ and $y = 2t - t^2$, for $t \in \mathbb{R}$, show that $\frac{dy}{dx}$ is constant.

(b) A curve is defined by the equation $x^2 - 2xy + 3y^2 + 4y = 22$.

(i) Find $\frac{dy}{dx}$ in terms of x and y.

(ii) The points $(-3, 1)$ and $(1, -3)$ are both on this curve.
Show that the tangents at these two points are parallel to each other.

(c) Let $f(x) = 32x^3 - 48x^2 + 20x - 1$, where $x \in \mathbb{R}$.

(i) Show that f has a root between 0 and 1.

(ii) Take $x_1 = 0{\cdot}5$ as a first approximation to this root. Use the Newton-Raphson method to find x_2 and x_3, the second and third approximations.

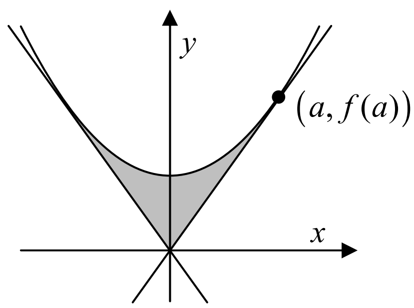
(iii) What can you conclude about all further approximations?

8. **(a)** Find $\int \left(1 + \cos 2x + e^{3x}\right) dx$.

(b) **(i)** Evaluate $\int_1^3 \frac{12}{3x-2}\, dx$.

(ii) Evaluate $\int_0^{\frac{\pi}{8}} \sin^2 2x \, dx$.

(c) The function f is given by $f(x) = x^2 + k$, where k is a positive constant.



(i) The tangent to the curve $y = f(x)$ at the point $(a, f(a))$ passes through the origin, where $a > 0$.
Express a in terms of k.

(ii) The tangent at $(-a, f(-a))$ also passes through the origin. Find, in terms of k, the area of the region enclosed by these two tangents and the curve.

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