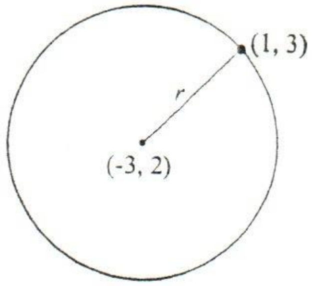
5^{th} Year - Hons Maths   Problem Set 7 [The Circle] - Solutions

Q1 [2008]

(a) A circle with centre $(-3, 2)$ passes through the point $(1, 3)$. Find the equation of the circle.



Circle C with centre (h, k), radius r.

$$(x-h)^2+(y-k)^2=r^2 \quad \text{....... 2}$$

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \quad \text{....... 1}$$

$r=\sqrt{(-3-1)^2+(2-3)^2}$

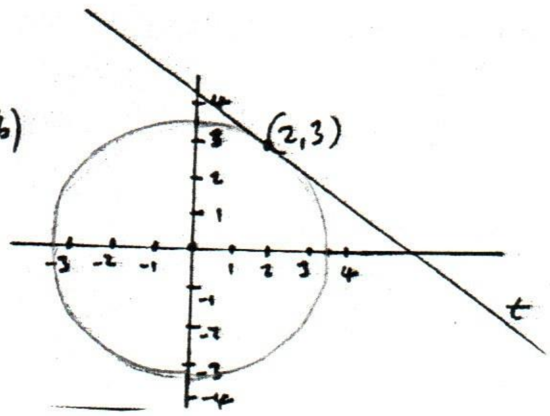
$\Rightarrow r=\sqrt{16+1}=\sqrt{17}$

Equation of the circle: $(x-(-3))^2+(y-2)^2=(\sqrt{17})^2$

$\therefore (x+3)^2+(y-2)^2=17$

(b) A tangent is drawn to the circle $x^2+y^2=13$ at the point $(2, 3)$. This tangent crosses the x-axis at $(k,0)$. Find the value of k.

1(b)



slope of $r = \frac{3-0}{2-0} = \frac{3}{2}$

slope of $t = \frac{-2}{3}$

Eqn of t: $y-3 = \frac{-2}{3}(x-2)$

$3(y-3) = -2(x-2)$

$3y-9 = -2x+4$

Eqn of t: $2x+3y-13=0$

At x-axis, y-value $= 0$

$2x+3(0)-13=0$

$2x = 13$

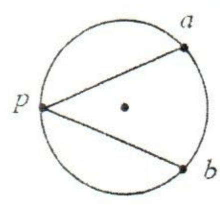
$x = \frac{13}{2}$

$(k, 0) = \left(\frac{13}{2}, 0\right)$

(c) A circle passes through the points $A(8, 5)$ and $B(9, -2)$. The centre of the circle lies on the line $2x - 3y - 7 = 0$.

(i) Find the equation of the circle.

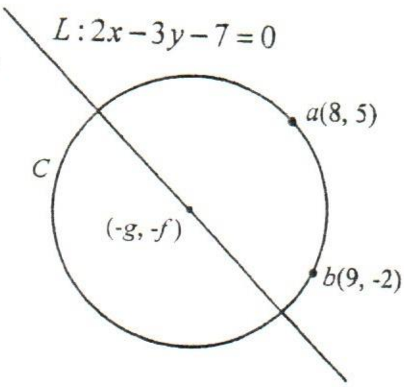
(ii) P is a point on the major arc ab of the circle. Show that $|\angle apb| = 45°$



1 (c) (i)

$x^2 + y^2 + 2gx + 2fy + c = 0$ **3**

Substitute points a and b into the equation of the circle, C.



$a \in C \Rightarrow 64 + 25 + 16g + 10f + c = 0$

$\therefore 16g + 10f + c = -89$.....**(1)**

$b \in C \Rightarrow 81 + 4 + 18g - 4f + c = 0$

$\therefore 18g - 4f + c = -85$.....**(2)**

$(-g, -f) \in L \Rightarrow -2g + 3f - 7 = 0$

$\therefore 2g - 3f = -7$....**(3)**

Eliminate c by subtracting Eqn. **(1)** from Eqn. **(2)**:

$\therefore 2g - 14f = 4$.....**(4)**

Now subtract Eqn. **(4)** from Eqn. **(3)**:

$\therefore 11f = -11 \Rightarrow f = -1$

Substitute this value of f into Eqn. **(3)**:

$\therefore 2g - 3(-1) = -7 \Rightarrow 2g = -10$

$\therefore g = -5$

Substitute these values of g and f into Eqn. **(1)**:

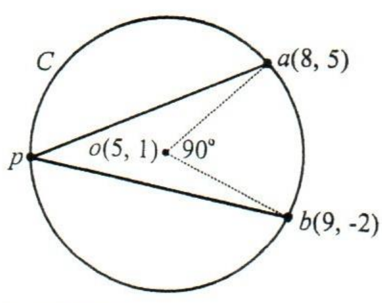
$\therefore 16(-5) + 10(-1) + c = -89 \Rightarrow -80 - 10 + c = -89$

$\therefore c = 1$

Equation of C: $x^2 + y^2 - 10x - 2y + 1 = 0$

1 (c) (ii)

If $|\angle apb| = 45°$ then the angle standing at the centre must be 90°. Therefore, you need to show that ao is perpendicular to bo, where o is the centre of the circle.



Slope of ao: $m_1 = \dfrac{5-1}{8-5} = \dfrac{4}{3}$

Slope of bo: $m_2 = \dfrac{-2-1}{9-5} = -\dfrac{3}{4}$

The slopes are perpendicular as $m_1 \times m_2 = -1$.

$m = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$ **2**

Q2 [2007]

(a) $x^2 + y^2 - 4x - 6y + 5 = 0$ and $x^2 + y^2 - 6x - 8y + 23 = 0$ are two circles.

(i) Prove that the circles touch internally.

(ii) Find the coordinates of the point of contact of the two circles.

Circle C with centre $(-g, -f)$, radius r.

$x^2 + y^2 + 2gx + 2fy + c = 0$ **3**

$r = \sqrt{g^2 + f^2 - c}$ **4**

C_1: $x^2 + y^2 - 4x - 6y + 5 = 0$

Centre $p_1(2, 3)$, $r_1 = \sqrt{4+9-5} = \sqrt{8} = 2\sqrt{2}$

C_2: $x^2 + y^2 - 6x - 8y + 23 = 0$

Centre $p_2(3, 4)$, $r_2 = \sqrt{9+16-23} = \sqrt{2}$

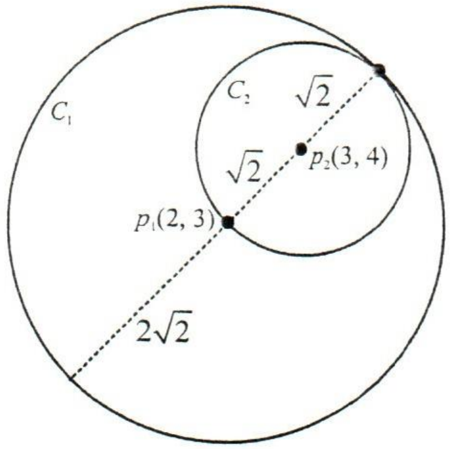
(a) (i)

INTERNAL TOUCH $|p_1p_2| = r_1 - r_2$

$|p_1p_2| = \sqrt{(2-3)^2 + (3-4)^2} = \sqrt{2}$

$r_1 - r_2 = 2\sqrt{2} - \sqrt{2} = \sqrt{2}$

Therefore, the circles touch internally.



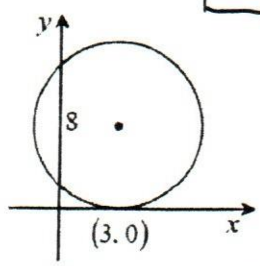
(a) (ii)

As can be seen from the diagram, the centre of C_1 lies on C_2 because its radius is twice that of C_2. The point (3, 4) is the mid-point of (2, 3) and the point of contact.

$(2,3) \rightarrow (3,4) \rightarrow (4,5)$

(4, 5) is the point of contact between the two circles.

(b) A circle has its centre in the first quadrant. The x-axis is a tangent to the circle at the point (3, 0). The circle cuts the y-axis at points that are 8 units apart. Find the equation of the circle.



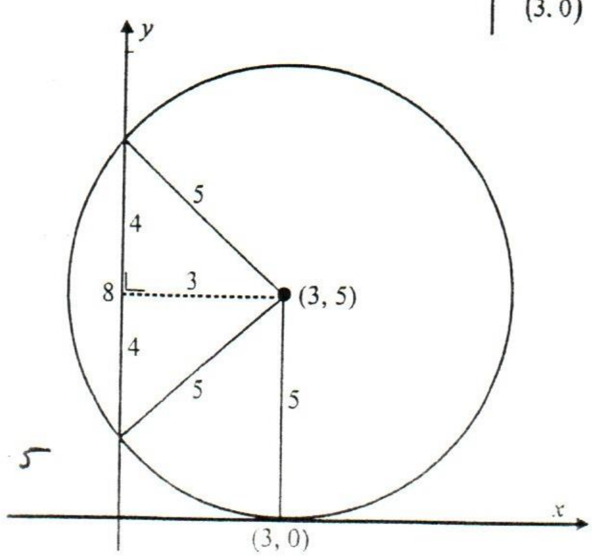
Centre (3, 5), $r = 5$

Circle C with centre (h, k), radius r.

$(x-h)^2 + (y-k)^2 = r^2$ **2**

Eqn. of circle: $(x-3)^2 + (y-5)^2 = 25$

Note: Because circle is tangent to the x-axis $\Rightarrow$ centre (3, 5) where r 5



Q3

2006

(a) $a(-1, -3)$ and $b(3, 1)$ are the end-points of a diameter of a circle. Write down the equation of a circle.

SOLUTION

(a)

The centre o is the mid-point of $[ab]$.

$$\text{Mid-point} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-1+3}{2}, \frac{-3+1}{2}\right) = (1, -1)$$

The radius of the circle is half the distance $|ab|$.

$$r = \tfrac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \tfrac{1}{2}\sqrt{(3+1)^2+(1+3)^2} = \tfrac{1}{2}\sqrt{32} = 2\sqrt{2}$$

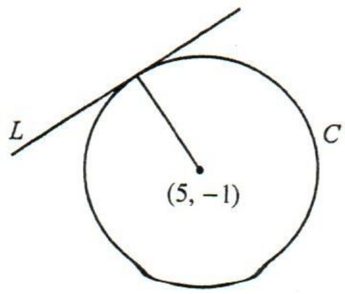
Circle C with centre (h, k), radius r. $\boxed{(x-h)^2+(y-k)^2=r^2}$ 2

C: $(x-1)^2+(y+1)^2=(2\sqrt{2})^2 \Rightarrow (x-1)^2+(y+1)^2=8$. This answer is fine. However, if you decide to expand the equation you will get: $x^2+y^2-2x+2y-6=0$

(b) Circle C has centre $(5, -1)$. The line L: $3x-4y+11=0$ is a tangent to C.

(i) Show that the radius of C is 6.

(ii) The line $x+py+1=0$ is also a tangent to C. Find two possible values of p.



(b) (i)

The radius of the circle is the perpendicular distance from the centre to the tangent.

Centre $(5, -1)$, L: $3x-4y+11=0$

$$d = \frac{|3(5)-4(-1)+11|}{\sqrt{3^2+(-4)^2}} = \frac{|15+4+11|}{\sqrt{25}} = \frac{30}{5} = 6$$

$\boxed{d = \dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}}$ 8

(b) (ii)

The perpendicular distance of the centre to this line is the radius (6 units).

Centre $(5, -1)$, L: $x+py+1=0$, $d=r=6$

$$\therefore 6 = \frac{|5+p(-1)+1|}{\sqrt{1^2+p^2}} \Rightarrow 6\sqrt{p^2+1} = |6-p|$$

$\boxed{d = \dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}}$ 8

$\Rightarrow 36(p^2+1) = 36-12p+p^2$ [Square both sides]

$\Rightarrow 36p^2+36 = 36-12p+p^2 \Rightarrow 35p^2+12p=0$

$\Rightarrow p(35p+12)=0 \Rightarrow p = 0, -\frac{12}{35}$

(c) The circle s is $x^2 + y^2 + 4x + 4y - 17 = 0$ and k is the line $4x + 3y = 12$.

(i) Show that the line k does not intersect s.

(ii) Find the co-ordinates of the point on s that is closest to k.

(c) (i)

To show that the line K does not intersect the circle S can be done in two ways:

Method 1: Solve K and S simultaneously and show it has no real solutions.

Method 2: Show that the perpendicular distance from the centre of the circle to the line K is greater than the radius of the circle. [This is a better method.]

Method 1: K: $4x+3y=12 \Rightarrow x=\dfrac{12-3y}{4}$

$$S:\ x^2+y^2+4x+4y-17=0 \Rightarrow \left(\frac{12-3y}{4}\right)^2+y^2+4\left(\frac{12-3y}{4}\right)+4y-17=0$$

$$\Rightarrow \left(\frac{144-72y+9y^2}{16}\right)+y^2+12-3y+4y-17=0$$

$$\Rightarrow 144-72y+9y^2+16y^2+192-48y+64y-272=0$$

$$\Rightarrow 25y^2-56y+64=0$$

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

.... 4

Remember: If $b^2-4ac \geq 0 \Rightarrow$ Real roots.
If $b^2-4ac<0 \Rightarrow$ Unreal or complex roots.

$a=25,\ b=-56,\ c=64$

$b^2-4ac=(-56)^2-4(25)(64)=3136-6400=-3264<0$

Therefore, there are no real solutions and so K and S do not intersect.

Method 2:

S: $x^2+y^2+4x+4y-17=0$

Centre $(-2,-2)$, $r=\sqrt{g^2+f^2-c}=\sqrt{4+4+17}=5$

$$d=\frac{|4(-2)+3(-2)-12|}{\sqrt{4^2+3^2}}=\frac{|-26|}{5}=\frac{26}{5}>5$$

Circle C centre $(-g,-f)$, radius r.

$x^2+y^2+2gx+2fy+c=0$

.... 3

$r=\sqrt{g^2+f^2-c}$

.... 4

$d=\dfrac{\lvert ax_1+by_1+c\rvert}{\sqrt{a^2+b^2}}$

.... 8

[2004]

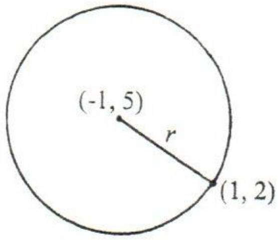
(a) A circle has centre $(-1, 5)$ and passes through the point $(1, 2)$. Find the equation of the circle.

Solution

(a)

Circle C with centre (h, k), radius r.

$(x-h)^2+(y-k)^2=r^2$ **2**



Centre $(-1, 5)$, $r=\sqrt{(-1-1)^2+(5-2)^2}=\sqrt{4+9}=\sqrt{13}$

Circle: $(x+1)^2+(y-5)^2=13$

Multiplying this equation also gives $x^2+y^2+2x-10y+13=0=13$

(b) The point $a(5, 2)$ is on the circle K: $x^2+y^2+px-2y+5=0$.

(i) Find the value of p.

(ii) The line L: $x-y-1=0$ intersects the circle K. Find the co-ordinates of the points of intersection.

(b) (i)

If a point is on the circle you can substitute it into the circle equation.

$\therefore 25+4+5p-4+5=0 \Rightarrow 5p=-30 \Rightarrow p=-6$

(b) (ii)

Steps

1. Isolate x or y using equation of the line.
2. Substitute into the equation of the circle and solve simultaneously.

1. L: $x-y-1=0 \Rightarrow x=y+1$

2. K: $x^2+y^2-6x-2y+5=0 \Rightarrow (y+1)^2+y^2-6(y+1)-2y+5=0$

$\Rightarrow y^2+2y+1+y^2-6y-6-2y+5=0 \Rightarrow 2y^2-6y=0$

$\Rightarrow y^2-3y=0 \Rightarrow y(y-3)=0 \Rightarrow y=0, 3 \Rightarrow x=1, 4$

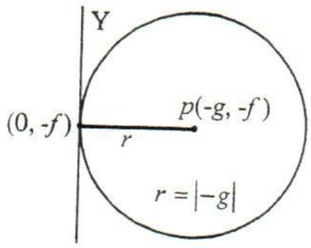
Ans: Points of intersection are $(1, 0)$ and $(4, 3)$.

(c) The y-axis is a tangent to the circle $x^2+y^2+2gx+2fy+c=0$.

(i) Prove that $f^2=c$.

(ii) Find the equations of the circles that pass through the points $(-3, 6)$ and $(-6, 3)$ and have the y-axis as a tangent.

(c) (i)



$\Rightarrow r^2=g^2=g^2+f^2-c$

$\Rightarrow \boxed{c=f^2}$

(c) (ii)

$(-3, 6)$ is on the circle $x^2+y^2+2gx+2fy+c=0$

$\Rightarrow 9+36-6g+12f+c=0 \Rightarrow 6g-12f-c=45$...**(1)**

$(-6, 3)$ is on the circle $x^2+y^2+2gx+2fy+c=0$

$\Rightarrow 36+9-12g+6f+c=0 \Rightarrow 12g-6f-c=45$...**(2)**

Y-axis is a tangent $\Rightarrow c=f^2$...**(3)**

Now look at the three equations. Eliminate g from equations **(1)** and **(2)**.

$12g-6f-c=45$...**(2)**	
$6g-12f-c=45$...**(1)** $\times(-2)$	

$\rightarrow$

$$\begin{array}{r} 12g-6f-c=45 \\ -12g+24f+2c=-90 \\ \hline 18f+c=-45 \end{array}$$ ....**(4)**

Substitute equation **3** into **4**.

$18f+c=-45 \Rightarrow 18f+f^2=-45 \Rightarrow f^2+18f+45=0$

$\Rightarrow (f+15)(f+3)=0 \Rightarrow f=-15, -3$

Using equation **3**: $c=f^2 \Rightarrow c=225, 9$

Using equation **1**: $6g-12f-c=45 \Rightarrow g=\dfrac{45+12f+c}{6} \Rightarrow g=15, 3$

Therefore the two equations are:

$x^2+y^2+6x-6y+9=0$ and $x^2+y^2+30x-30y+225=0$.