AN ROINN OIDEACHAIS AGUS EOLAÍOCHTA

DEPARTMENT OF EDUCATION AND SCIENCE

Scéim Mharcála *Scrúduithe Ardteistiméireachta, 2002*

Matamaitic *Ardleibhéal*

Marking Scheme *Leaving Certificate Examination, 2002*

Mathematics *Higher Level*

An Roinn Oideachais agus Eolaíochta

Leaving Certificate Examination 2002

Marking Scheme

MATHEMATICS

Higher Level

Paper 1

General Instructions

Penalties are applied as follows:

numerical slips, misreadings	(−1) each
blunders, major omissions	(−3) each.

Note 1: The lists of slips, blunders and attempts given in the marking scheme are not exhaustive.

Note 2: A serious blunder, omission or misreading merits the attempt mark at most.

Note 3: The attempt mark (Att) for a section is the final mark for that section. Where deductions result in a mark that is less than the attempt mark, the attempt mark is awarded.

Note 4: Particular cases and verifications are, in general, awarded the attempt mark only.

Note 5: All of the candidate's work, including any that is cancelled, is marked and the highest scoring solutions are allowed.

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QUESTION 1

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**
Part (c)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

1(a) Solve the equation

$$x = \sqrt{x+2}\,.$$

Quadratic	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

1(a)

$$x = \sqrt{x+2}$$
$$x^2 = x+2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \quad \text{or} \quad x = -1$$

Check $x = 2$: LHS = 2 RHS $= \sqrt{2+2} = 2$ …….. LHS = RHS

$x = -1$: LHS = -1 RHS = $\sqrt{-1+2} = \sqrt{1} = 1$ ……. LHS ≠ RHS

$\Rightarrow \quad x = 2$

Blunders (–3)

B1 Indices.

B2 Factors (once only).

B3 Root formula (once only).

B4 Deduction of values from factors or no value.

Slips (–1)

S1 Numerical.

S2 Extra value.

Attempts

A1 $x = 2$ and no other work merits 2 marks.

Part (b)	20 (5, 5, 10) marks	Att (2, 2, 3)

1(b) The cubic equation $x^3 - 4x^2 + 9x - 10 = 0$ has one integer root and two complex roots. Find the three roots.

Linear Factor	**5 marks**	**Att 2**
Quadratic Factor	**5 marks**	**Att 2**
Roots	**10 marks**	**Att 3**

1(b)

$$f(x) = x^3 - 4x^2 + 9x - 10 = 0$$
$$f(1) = 1 - 4 + 9 - 10 \neq 0$$
$$f(2) = 8 - 16 + 18 - 10 = 0 \Rightarrow (x-2) \text{ is factor}$$

$$\begin{array}{r|l} & x^2 - 2x + 5 \\ x-2 & x^3 - 4x^2 + 9x - 10 \\ & \underline{x^3 - 2x^2} \\ & \quad -2x^2 + 9x \\ & \quad \underline{-2x^2 + 4x} \\ & \qquad 5x - 10 \\ & \qquad \underline{5x - 10} \end{array}$$

$$f(x) = 0 \quad \Rightarrow (x-2)(x^2 - 2x + 5) = 0$$
$$\Rightarrow x - 2 = 0 \quad \text{or} \quad x^2 - 2x + 5 = 0$$
$$\Rightarrow x = 2 \quad \text{or} \quad x = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Roots are 2, $(1 + 2i)$, $(1 - 2i)$

Blunders (–3)

B1 Test for root.
B2 Deduction of factor from root.
B3 Indices.
B4 Root formula (once only).
B5 Deduction of root from factor or no deduction.

Slips (–1)

S1 Numerical.
S2 Not changing sign when subtracting in division.

Worthless

W1 $x(x^2 - 4x + 9) = 10$, with or without further work.

Note If there is a remainder after division or incomplete division, candidate can only get Att, at most, for roots.

Part (c)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**

1(c) $(p+r-t)x^2+2rx+(t+r-p)=0$ is a quadratic equation, where *p, r,* and *t* are integers.

Show that

(i) the roots are rational

(ii) one of the roots is an integer.

b^2-4ac	**10 marks**	**Att 3**
Perfect square	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

1(c)

$$f(x)=(p+r-t)x^2+2rx+(t+r-p)=0$$

$$[r-(t-p)]x^2+2rx+[r+(t-p)]=0$$

Let $(t-p)=k$

$$\Rightarrow (r-k)x^2+2rx+(r+k)=0$$

In applying quadratic formula, $b^2-4ac=(2r)^2-4(r-k)(r+k)$

$$= 4r^2-4(r^2-k^2)$$

$$= 4r^2-4r^2+4k^2$$

$$= 4k^2$$

$$=[2(t-p)]^2 \Rightarrow \quad \text{perfect square}$$

$\Rightarrow$ quadratic has rational roots

Roots: $$x=\frac{-2r\pm\sqrt{[2(t-p)]^2}}{2(p+r-t)}=\frac{-2r\pm 2(t-p)}{2(p+r-t)}=\frac{-r\pm(t-p)}{p+r-t}$$

$$\Rightarrow x=\frac{-r+t-p}{p+r-t}=\frac{-(p+r-t)}{p+r-t}=-1 \text{ (integer) or } x=\frac{-r-t+p}{p+r-t}$$

$\Rightarrow$ the roots are -1 and $\dfrac{p-r-t}{p+r-t}$.

or

1(c)

$f(x)=(r-a)x^2+2rx+(r+a)$ where $a=t-p$

$f(0)=0+0+(r+a)\neq 0$

$f(1)=(r-a)+2r+(r+a)\neq 0$

$f(-1)=r-a-2r+r+a=0$

$\Rightarrow x=(-1)$ is a root of $f(x)=0 \Rightarrow$ one root is an integer.

Let $\alpha=$ other root

Product of roots $=\dfrac{r+a}{r-a}=(\alpha)(-1)$

$$\Rightarrow \alpha=\frac{-r-a}{r-a}=\frac{-r-t+p}{r-t+p} \quad \ldots\ldots \text{ other root.}$$

or

1(c)

$$\left[(p+r-t)x+(t+r-p)\right]\left[x+1\right]=0$$

$$x=-1 \qquad \text{or} \qquad x=\frac{-t-r+p}{p+r-t}$$

Blunders (–3)
B1 Indices.
B2 Root formula (once only).
B3 Test for root.
B4 Sum or product of roots.
B5 Factors (once only).

Worthless
W1 When (i) treated as sum and product of roots.

Note Cannot get marks for finish if not perfect square.

QUESTION 2

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**
Part (c)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**

Part (a)	**10 (5, 5)marks**	**Att (2, 2)**

2(a)

Solve, without using a calculator, the following simultaneous equations:

$$x + 2y + 4z = 7$$
$$x + 3y + 2z = 1$$
$$-y + 3z = 8.$$

Elimination *x*	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

2(a)

(i) $x + 2y + 4z = 7$

(ii) $x + 3y + 2z = 1$

(iii) $-y + 3z = 8$

(i) $x + 2y + 4z = 7$

(ii) $x + 3y + 2z = 1$

$-y + 2z = 6$ …(iv)

(iv) $-y + 2z = 6$

(iii) $-y + 3z = 8$

$-z = -2$

$z = 2$

(iii) $-y + 3z = 8$

$-y + 6 = 8$

$-y = 2$

$y = -2$

(i) $x + 2y + 4z = 7$

$x - 4 + 8 = 7$

$x = 3$

$x = 3$

$y = -2$

$z = 2$

Blunders (–3)

B1 Not finding 2nd unknown or 3rd unknown (having found 1st).

B2 Multiplying one side of equation only.

Slips (–1)

S1 Numerical.

Worthless (0)

W1 Trial and error.

Note If *y* or *z* eliminated, must do two cancellations for 5 marks.

Part (b)(i)	**10 (5, 5) marks**	**Att (2, 2)**

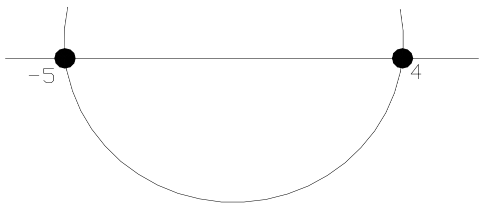
2(b)(i) Find the range of values of $x \in \mathbf{R}$ for which

$$x^2 + x - 20 \le 0.$$

Roots/complete square	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

2(b)(i)

$$x^2 + x - 20 \le 0$$
$$(x+5)(x-4) = 0$$
$$x = -5 \quad \text{or} \quad x = 4$$



$$-5 \le x \le 4$$

or

2(b)(i)

$$x^2 + x \le 20$$
$$x^2 + x + \frac{1}{4} \le 20\frac{1}{4}$$
$$\left(x + \frac{1}{2}\right)^2 \le \frac{81}{4}$$
$$\left(x + \frac{1}{2}\right)^2 \le \left(\frac{9}{2}\right)^2$$
$$\Rightarrow \quad -\frac{9}{2} \le \left(x + \frac{1}{2}\right) \le \frac{9}{2}$$
$$-5 \le x \le 4$$

Blunders (–3)

B1 Inequality sign.
B2 Indices.
B3 Factors (once only).
B4 Root formula (once only).
B5 Deduction root from factor.
B6 Range not stated.
B7 Incorrect range.
B8 Completing square.

Slips (–1)

S1 Numerical.

Part (b)(ii) **10 marks** **Att 3**

(ii)	Let $g(x) = x^n + 3$, for all $x \in \mathbf{R}$, where $n \in \mathbf{N}$. Show that if n is odd then $g(x) + g(-x)$ is constant.

Part (b)(ii) **10 marks** **Att 3**

2(b)(ii)

$$g(x) = x^n + 3 \qquad n \text{ odd}$$
$$g(-x) = (-x)^n + 3 = -x^n + 3$$
$$g(x) + g(-x) = (x^n + 3) + (-x^n + 3)$$
$$= 6$$

Blunders (–3)
B1 Indices.

Slips (–1)
S1 Numerical.

Attempts
A1 Particular odd value of *n*.
A2 *x* in answer.

Worthless (0)
W1 Particular values of *x*.

Part (c) **20 (10, 5, 5) marks** **Att (3, 2, 2)**

2(c)(i) Show that if the roots of $x^2 + bx + c = 0$ differ by 1, then $b^2 - 4c = 1$.

(ii) The roots of the equation $x^2 + (4k - 5)x + k = 0$ are consecutive integers. Using the result from part **(i)**, or otherwise, find the value of *k* and the roots of the equation.

$b^2 - 4c = 1$	**10 marks**	**Att 3**
Values *k*	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

2(c)(i)

$$x^2 + bx + c = 0$$

$$\text{Roots } = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

$$\Rightarrow \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4c}}{2} \quad \text{are roots}$$

$$x_1 - x_2 = -\frac{b}{2} + \frac{\sqrt{b^2 - 4c}}{2} + \frac{b}{2} + \frac{\sqrt{b^2 - 4c}}{2}$$

$$\Rightarrow \quad 1 = \sqrt{b^2 - 4c}$$

$$\Rightarrow \quad 1 = b^2 - 4c$$

or

2(c)(i)

Let α and $(\alpha + 1)$ be the two consecutive roots

$$x^2 - (-b)x + (c) = 0$$

$$x^2 - [2\alpha + 1]x + [\alpha(\alpha + 1)] = 0$$

$$\Rightarrow b = -(2\alpha + 1) \text{ and } c = \alpha^2 + \alpha$$

$$b^2 - 4c = [-(2\alpha + 1)]^2 - 4(\alpha^2 + \alpha)$$

$$= 4\alpha^2 + 4\alpha + 1 - 4\alpha^2 - 4\alpha$$

$$\Rightarrow \; b^2 - 4c = 1$$

2(c)(ii)

$$x^2 + (4k - 5)x + k = 0$$

$$x^2 + bx + c = 0$$

$$\Rightarrow b = (4k - 5) \text{ and } c = k$$

From **(i)**, $b^2 - 4c = 1 \Rightarrow (4k - 5)^2 - (4k) = 1$

$$16k^2 - 40k + 25 - 4k - 1 = 0$$

$$16k^2 - 44k + 24 = 0$$

$$4k^2 - 11k + 6 = 0$$

$$(4k - 3)(k - 2) = 0$$

$$k = \frac{3}{4} \text{ or } k = 2$$

$k = 2: \; x^2 + 3x + 2 = 0$

$$(x + 1)(x + 2) = 0$$

$x = -1$ or $x = -2$ $\Rightarrow$ $k = 2$ is required value and roots are -1 and -2.

$k = \frac{3}{4}: x^2 - 2x + \frac{3}{4} = 0$ does not have integral roots $\Rightarrow k \neq \frac{3}{4}$.

Blunders (–3)

B1 Indices.
B2 Roots not α and $\alpha \pm 1$.
B3 Statement of quadratic equation.
B4 Not like to like.
B5 Expansion $(m+n)^2$.
B6 Factors (once only).
B7 Root formula (once only).
B8 Deduction of value from factor, or no value from factor.
B9 Extra value of *k* only.
B10 Not finding roots.

Slips (–1)

S1 Numerical.

QUESTION 3

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 10, 5) marks**	**Att (2, 3, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

3(a)

Express $-1+\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$.

r	**5 marks**	**Att 2**
θ	**5 marks**	**Att 2**

3(a)

$$z = -1 + i\sqrt{3}$$

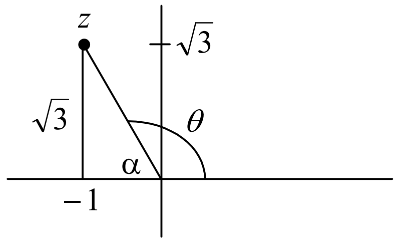
$$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4}$$

$$r = 2$$

$$\tan\alpha = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\Rightarrow \alpha = 60° \quad \text{and} \quad \theta = \frac{2\pi}{3}$$

$$z = 2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right] = 2(\cos 120^\circ + i\sin 120^\circ)$$



Blunders (–3)
B1 Argument.
B2 Modulus.
B3 Trigonometric definition.
B4 Indices.
B5 *i.*

Slips (-1)
S1 Numerical.
S2 Trigonometric values.

Part (b)(i)	10 (5, 5) marks	Att (2, 2)

3(b)(i) Given that $z = 2 - i\sqrt{3}$, find the real number t such that $z^2 + tz$ is real.

$\left(z^2 + tz\right)$	5 marks	Att 2
value ***t***	**5 marks**	**Att 2**

3(b)(i)

$$z = 2 - i\sqrt{3}$$

$$z^2 = \left(2 - i\sqrt{3}\right)^2 = 4 - 4\sqrt{3}i - 3 = 1 - \left(4\sqrt{3}\right)i$$

$$z^2 + tz = \left[1 - i\left(4\sqrt{3}\right)\right] + t\left[2 - i\sqrt{3}\right] = k + (0)i$$

$$(1 + 2t) + \left[-4\sqrt{3} - t\sqrt{3}\right]i = k + (0)i$$

$$\Rightarrow -4\sqrt{3} - t\sqrt{3} = 0$$

$$t = -4$$

Blunders (–3)

B1 i

B2 $(a+b)^2$.

B3 Indices.

B4 Not real to real etc.

Part (b)(ii)	10 (5, 5) marks	Att (2, 2)

3(b)(ii) w is a complex number such that

$$w\bar{w} - 2iw = 7 - 4i,$$

where $\bar{w}$ is the complex conjugate of w.

Find the two possible values of w.
Express each in the form $p + qi$, where $p, q \in R$.

p	5 marks	Att 2
Express	**5 marks**	**Att 2**

3(b)(ii)

$$w\bar{w} - 2iw = 7 - 4i$$

$$w = p + qi$$

$$(p + qi).(p - qi) - 2i(p + qi) = 7 + (-4)i$$

$$p^2 + q^2 - 2pi + 2q = 7 + (-4)i$$

$$\left(p^2 + q^2 + 2q\right) + (-2p)i = 7 + (-4)i$$

$$\Rightarrow -2p = -4$$

$$p = 2$$

$$p^2 + q^2 + 2q = 7$$

$$4 + q^2 + 2q = 7$$

$$q^2 + 2q - 3 = 0$$

$$(q + 3)(q - 1) = 0$$

$q = -3$ or $q = 1$ $p + qi = 2 - 3i$ or $2 + i$.

Blunders (–3)
B1 Conjugate.
B2 *i*
B3 Indices.
B4 Not real to real etc.
B5 Factors once only.
B6 Root formula once only.
B7 Value from factor.
B8 Not in correct form (once only).

Slips (–1)
S1 One value of *w* only.

Worthless
W1 $\overline{w}$ not a complex number.

Part (c)	**20 (5, 10, 5) marks**	**Att (2, 3, 2)**

3(c) The following three statements are true whenever x and y are real numbers:

- $x + y = y + x$
- $xy = yx$
- If $xy = 0$ then either $x = 0$ or $y = 0$.

Investigate whether the statements are also true when x is the matrix $\begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}$ and y is the matrix $\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$.

x* + *y	**5 marks**	**Att 2**
xy* = *yx	**10 marks**	**Att 3**
***xy* = 0**	**5 marks**	**Att 2**

3(c) $x = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}$ $y = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$

(i) $x + y = \begin{pmatrix} 5 & 2 \\ 0 & 11 \end{pmatrix}$

$y + x = \begin{pmatrix} 5 & 2 \\ 0 & 11 \end{pmatrix} \Rightarrow x + y = y + x \Rightarrow$ true

(ii) $x.y = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$y.x = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix} = \begin{pmatrix} -12 & 4 \\ -36 & 12 \end{pmatrix}$

$x.y \neq y.x \Rightarrow$ false

(iii) $x.y = 0$ but $x \neq 0$ and $y \neq 0 \Rightarrow$ false

Blunders (–3)
B1 Incorrect deduction or no deduction.

Slips (–1)
S1 Each incorrect element matrix.
S2 Numerical.

Worthless
W1 Incorrect deduction and no work in $xy = 0$.

QUESTION 4

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 10, 5) marks**	**Att (2, 3, 2)**
Part (c)	**20 (5, 10, 5) marks**	**Att (2, 3, 2)**

Part (a) **10 marks** **Att 3**

4(a)

Find, in terms of *n,* the sum of the first *n* terms of the geometric series

$$3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\ldots$$

$\boldsymbol{S_n}$ **of G.P** **10 marks** **Att 3**

4(a)

$$S_n = 3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\ldots\ldots\ldots+\frac{3}{2^{n-1}}$$

$$G.P : a = 3 \qquad r=\frac{1}{2}$$

$$S_n = \frac{a\left(1-r^n\right)}{\left(1-r\right)} = \frac{3\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} = 6\left[1-\left(\frac{1}{2}\right)^n\right]$$

Blunders (–3)
B1 Formula S_n .
B2 Incorrect *'a'*.
B3 Incorrect *'r'*.
B4 Incorrect number of terms.
B5 Indices.

Slips (–1)
S1 Numerical.

Attempts
A1 S_∞

Worthless (0)
W1 Sum of *A.P.*

Part (b)	20 (5, 10, 5) marks	Att (2, 3, 2)

4(b) **(i)** Show that

$$\frac{2}{k(k+2)} = \frac{1}{k} - \frac{1}{k+2} \text{ for all } k \in \mathbf{R},\ k \neq 0, -2.$$

(ii) Evaluate, in terms of *n*, $\sum_{k=1}^{n} \frac{2}{k(k+2)}$.

(iii) Evaluate $\sum_{k=1}^{\infty} \frac{2}{k(k+2)}$.

(i)	**5 marks**	**Att 2**
(ii)	**10 marks**	**Att 3**
(iii)	**5 marks**	**Att 2**

4(b)(i)

To prove $\frac{2}{k(k+2)} = \frac{1}{k} - \frac{1}{k+2}$ …….

LHS = $\frac{1}{k} - \frac{1}{k+2} = \frac{(k+2)-k}{k(k+2)} = \frac{2}{k(k+2)}$ = RHS

or

4(b)(i)

$$\frac{2}{k(k+2)} = \frac{a}{k} + \frac{b}{k+2}$$
$$2 = a(k+2) + b(k)$$
$$(0)k + (2) = (a+b)k + (2a)$$

Since this is true for all real $k \Rightarrow a + b = 0$ and $2 = 2a$

$\Rightarrow \quad a = 1$

$a + b = 0$

$1 + b = 0$

$\Rightarrow \quad b = -1$

$\Rightarrow \quad \frac{2}{k(k+2)} = \frac{1}{k} - \frac{1}{k+2}$, as required.

4(b)(ii)

$$\sum_{k=1}^{n}\frac{2}{k(k+2)} = S_n = U_n + U_{n-1} + U_{n-2} + \quad \ldots \quad + U_1$$

$$U_n = \frac{2}{n(n+2)} \qquad = \qquad \frac{1}{n} - \frac{1}{n+2}$$

$$U_{n-1} = \frac{2}{(n-1)(n+1)} \qquad = \qquad \frac{1}{n-1} - \frac{1}{n+1}$$

$$U_{n-2} = \frac{2}{(n-2)(n)} \qquad = \qquad \frac{1}{n-2} - \frac{1}{n}$$

$$U_{n-3} = \frac{2}{(n-3)(n-1)} \qquad = \qquad \frac{1}{n-3} - \frac{1}{n-1}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$U_4 = \frac{2}{4.6} \qquad = \qquad \frac{1}{4} - \frac{1}{6}$$

$$U_3 = \frac{2}{3.5} \qquad = \qquad \frac{1}{3} - \frac{1}{5}$$

$$U_2 = \frac{2}{2.4} \qquad = \qquad \frac{1}{2} - \frac{1}{4}$$

$$U_1 = \frac{2}{1.3} \qquad = \qquad \frac{1}{1} - \frac{1}{3}$$

$$S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \qquad = \qquad \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

or

$$\sum_{k=1}^{n}\left[\frac{1}{k} - \frac{1}{k+2}\right] = \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k+2}$$

$$= \left[1 + \frac{1}{2} + \sum_{k=3}^{n}\frac{1}{k}\right] - \left[\sum_{k=3}^{n}\frac{1}{k} + \frac{1}{n+1} + \frac{1}{n+2}\right]$$

$$= \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

4(b)(iii) Taking the limit as $n \to \infty$ gives $\sum_{k=1}^{\infty} \frac{2}{k(k+2)} = \frac{3}{2}$.

Blunders (–3)

B1 Not like to like.
B2 Cancellation must be shown or implied.
B3 $k = 0$ or $k = -2$.
B4 Term, or terms, omitted.
B5 Blunder in deduction of S_∞ from S_n.
B6 S_k

Slips (–1)

S1 Numerical.

Note Must show 3 terms at start and 2 at finish or vice versa.

Part (c)	**20 (5, 10, 5)marks**	**Att (2, 3, 2)**

4(c) Three numbers are in arithmetic sequence.
Their sum is 27 and their product is 704.
Find the three numbers.

Equations	**5 marks**	**Att 2**
Quadratic	**10 marks**	**Att 3**
Numbers	**5 marks**	**Att 2**

4(c)

$(a-d), a, (a+d)$ are in arithmetic sequence

Sum: $3a = 27 \Rightarrow a = 9$

Product: $(a-d).(a).(a+d) = 704$

$$a(a^2 - d^2) = 704$$

$$9(81 - d^2) = 704$$

$$81 - d^2 = \frac{704}{9}$$

$$\frac{729 - 704}{9} = d^2$$

$$d^2 = \frac{25}{9} = \left(\frac{5}{3}\right)^2$$

$$d = \pm\frac{5}{3}$$

$\Rightarrow \frac{32}{3}, \; 9, \; \frac{22}{3}$ are the three numbers.

or

4(c) $a, (a+d), (a+2d)$ are in arithmetic sequence

Sum $3a + 3d = 27$

$a + d = 9 \qquad \Rightarrow d = 9 - a$

Product: $(a)(a+d)(a+2d) = 704$

$9a(a+2d) = 704$

$9(a)[a + 2(9-a)] = 704$

$9a(18-a) = 704$

$9a^2 - 162a + 704 = 0$

$(3a-22)(3a-32) = 0$

$\Rightarrow a = \frac{22}{3}$ or $a = \frac{32}{3}$

$d = 9 - a = \frac{27}{3} - \frac{22}{3} = \frac{5}{3}$ $\qquad$ $d = 9 - a = \frac{27}{3} - \frac{32}{3} = -\frac{5}{3}$

$\Rightarrow \frac{32}{3}, \; 9, \; \frac{22}{3}$ are the three numbers.

or

4(c) Let the three numbers be x, y, z

A.P: $z - y = y - x$

$x - 2y + z = 0$........................(i)

Sum: $x + y + z = 27$......................................(ii)

Product: $xyz = 704$

(i) $x - 2y + z = 0$

(ii) $x + y + z = 27$

$-3y = -27$

$y = 9$

(ii) $x + y + z = 27$

$x + z = 18 \qquad \Rightarrow z = (18 - x)$

(iii) $xyz = 704$

$9xz = 704$

$9x(18-x) = 704$

$9x^2 - 162x + 704 = 0$

Using factors …….. $(3x-22)(3x-32) = 0$

$x = \frac{22}{3}$ or $x = \frac{32}{3}$ etc………..

or

Using formula ……. $9x^2 - 162x + 704 = 0$

$$x = \frac{162 \pm \sqrt{(162)^2 - 25344}}{18} = \frac{162 \pm \sqrt{900}}{18} = \frac{162 \pm 30}{18}$$

$$= \frac{192}{18} \quad \text{or} \quad \frac{132}{18}$$

$$= \frac{32}{3} \quad \text{or} \quad \frac{22}{3} \quad \text{etc ……}$$

Blunders (–3)

B1 Indices.
B2 Statement of 3 consecutive terms.
B3 Factors (once only).
B4 Root formula (once only).
B5 Deduction of value from factor.

QUESTION 5

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

5(a) Find the value of x in each case:

(i) $\frac{8}{2^x} = 32$

(ii) $\log_9 x = \frac{3}{2}$.

(i)	**5 marks**	**Att 2**
(ii)	**5 marks**	**Att 2**

5(a)(i)

$$\frac{8}{2^x} = 32$$
$$2^{3-x} = 2^5$$
$$\Rightarrow 3 - x = 5$$
$$x = -2$$

5(a)(ii)

$$\log_9 x = \frac{3}{2}$$
$$x = (9)^{\frac{3}{2}} = 27$$

Blunders (–3)

B1 Indices.

B2 Logs.

Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

5(b)

The first three terms in the binomial expansion of $(1+ax)^n$ are $1 + 2x + \frac{7x^2}{4}$.

(i) Find the value of a and the value of n.

(ii) Hence, find the middle term in the expansion.

Expansion	**5 marks**	**Att 2**
Equation in one variable	**5 marks**	**Att 2**
Values	**5 marks**	**Att 2**
(ii)	**5 marks**	**Att 2**

5(b) $(1+ax)^n = 1+2x+\frac{7x^2}{4}+......$

$$= 1+\binom{n}{1}(ax)+\binom{n}{2}(ax)^2+.....$$

$$= 1+nax+\frac{n(n-1)}{2}\cdot a^2\cdot x^2+..... \Rightarrow na=2 \text{ and } \frac{n(n-1)\cdot a^2}{2}=\frac{7}{4}$$

(i) : $na=2 \Rightarrow a=\frac{2}{n}$

(ii): $\frac{n(n-1)\cdot a^2}{2}=\frac{7}{4}$

$$\frac{n(n-1)\cdot 4}{2\cdot n^2}=\frac{7}{4}$$

$$\frac{2(n-1)}{n}=\frac{7}{4}$$

$$8n-8=7n \qquad \Rightarrow \qquad n=8$$

From (i): $a=\frac{2}{n}=\frac{1}{4}\Rightarrow\left(1+\frac{x}{4}\right)^8$ is the expression that has been expanded

There are 9 terms in expansion of $\left(1+\frac{x}{4}\right)^8 \Rightarrow U_5$ is middle term

$$U_5=\binom{8}{4}\left(\frac{x}{4}\right)^4=\frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}\cdot\frac{x^4}{256}=\frac{35x^4}{128}$$

or

$$\left(1+\frac{x}{4}\right)^8=1+\binom{8}{1}(ax)+\binom{8}{2}(ax)^2+\binom{8}{3}(ax)^3+\binom{8}{4}(ax)^4+......$$

$$\Rightarrow \quad U_5=\binom{8}{4}\left(\frac{x}{4}\right)^4=\frac{35x^4}{128}$$

Blunders (–3)

B1 Binomial expansion (once only).

B2 Indices.

B3 Value of $\binom{n}{r}$ or no value of $\binom{n}{r}$.

B4 Not like to like.

B5 Incorrect middle term.

Attempts

A1 Correct trial and error.

Note When n odd accept either of the two middle terms.

Part (c)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**

5(c) Prove by induction that, for any positive integer n,

$$x + x^2 + x^3 + \ldots + x^n = \frac{x(x^n - 1)}{x-1} \text{ where } x \neq 1.$$

***P*(1)**	**5 marks**	**Att 2**
Assume	**5 marks**	**Att 2**
***P(*k + 1)**	**10 marks**	**Att 3**

5(c)(i)

Let $P(n)$ be that $x + x^2 + x^3 + \ldots + x^n = \dfrac{x(x^n - 1)}{x-1}$

Prove $P(1)$ true: $\dfrac{x(x^1 - 1)}{x-1} = \dfrac{x(x-1)}{x-1} = x$ where $x \neq 1$

$\Rightarrow$ $P(1)$ true

Assume $P(k)$ true, i.e. $x + x^2 + x^3 + x^4 + \ \ldots \ + x^k = \dfrac{x(x^k - 1)}{x-1}$ *

Prove that $P(k+1)$ is true, i.e. $x + x^2 + x^3 + x^4 + \ \ldots \ + x^k + x^{k+1} = \dfrac{x(x^{k+1} - 1)}{x-1}$

Proof: $x + x^2 + x^3 + x^4 + \ \ldots \ + x^k + x^{k+1} = \dfrac{x(x^k - 1)}{x-1} + x^{k+1}$ from *

$$= \frac{x^{k+1} - x + x^{k+1}(x-1)}{(x-1)}$$

$$= \frac{x^{k+1} - x + x^{k+2} - x^{k+1}}{x-1}$$

$$= \frac{x^{k+2} - x}{x-1}$$

$$= \frac{x(x^{k+1} - 1)}{(x-1)}$$

$\Rightarrow$ $P(k+1)$ true if $P(k)$ true

Since $P(1)$ true, then $P(2)$, $P(3)$, $P(4)$ …… true
$\Rightarrow$ $P(n)$ true for any positive integer n.

Blunders (–3)
B1 Indices.
B2 $n \neq 1$.

Attempts
A1 Particular values of x.

Note Must prove $n = 1$ (not sufficient to state $P(n)$ true for $n = 1$).

QUESTION 6

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (10, 10) marks**	**Att (3, 3)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 marks**	**Att 3**

6(a) Differentiate $(x^4+1)^5$ with respect to x.

Part (a)	**10 marks**	**Att 3**

6(a)

$$y=(x^4+1)^5$$

$$\frac{dy}{dx}=5(x^4+1)^4\cdot 4x^3=20x^3(x^4+1)^4$$

Blunders (–3)
B1 Differentiation.
B2 Indices.

Attempts
A1 Error in differentiation formula.

Worthless (0)
W1 Integration.

Note Simplification of derivative not required.

Part (b)(i)	**10 marks**	**Att 3**

6(b)(i) Prove, from first principles, the addition rule:
if $f(x) = u(x) + v(x)$ then $\frac{df}{dx}=\frac{du}{dx}+\frac{dv}{dx}$.

Part (b)(i)	**10 marks**	**Att 3**

6(b)(i)

$$f(x) = u(x) + v(x)$$

$$f(x+h)=u(x+h)+v(x+h)$$

$$f(x+h)-f(x)=[u(x+h)-u(x)]+[v(x+h)-v(x)]$$

$$\frac{f(x+h)-f(x)}{h}=\left[\frac{u(x+h)-u(x)}{h}\right]+\left[\frac{v(x+h)-v(x)}{h}\right]$$

$$\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]=\lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}\right]+\lim_{h\to 0}\left[\frac{v(x+h)-v(x)}{h}\right]$$

$$\frac{df}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

or

6(b)(i)	$y = u + v$ where $y = f(x)$, $u = u(x)$, $v = v(x)$

$$y + \Delta y = (u + \Delta u) + (v + \Delta v)$$
$$\Delta y = \Delta u + \Delta v$$
$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$
$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Blunders (–3)
B1 No limit shown or implied or no indication $\to 0$.

Attempts
A1 Some effort at $f(x+h)$.

Worthless
W1 No 1st principles.

Note Limit appearing once is acceptable.

Part (b)(ii) **10 marks** **Att 3**

6(b)(ii) Given $y = 2x - \sin 2x$, find $\frac{dy}{dx}$.
Give your answer in the form $k \sin^2 x$, where $k \in Z$.

Part (b)(ii) **10 marks** **Att 3**

6(b)(ii)
$$y = 2x - \sin 2x$$
$$\frac{dy}{dx} = 2 - 2\cos 2x$$
$$= 2(1 - \cos 2x)$$
$$= 2(2\sin^2 x)$$
$$= 4\sin^2 x$$

Blunders (–3)
B1 Differentiation.
B2 Trigonometric formula.
B3 Not in required form.

Attempts
A1 Error in differentiation formula.

Worthless (0)
W1 Integration.

Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

6(c) The function $f(x) = ax^3 + bx^2 + cx + d$ has a maximum point at (0, 4) and a point of inflection at (1, 0).

Find the values of *a, b, c* and *d.*

Using 1st given point	**5 marks**	**Att 2**
Using 2nd given point	**5 marks**	**Att 2**
Using $f'(x) = 0$	**5 marks**	**Att 2**
Using $f''(x) = 0$ and finish	**5 marks**	**Att 2**

6(c) $y = ax^3 + bx^2 + cx + d$

Max at $(0,4) \Rightarrow y = 4$ at $x = 0$ $\Rightarrow d = 4$..(i)

Point of inflection at $(1,0) \Rightarrow y = 0$ at $x = 1$

$0 = a + b + c + d \Rightarrow a + b + c = -4$......................................(ii)

Max at $x = 0 \Rightarrow \frac{dy}{dx} = 0$ at $x = 0$

$y = ax^3 + bx^2 + cx + d$

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

$3a(0) + 2b(0) + c = 0 \Rightarrow c = 0$...$(iii)$

Point of inflection at $x = 1 \Rightarrow \frac{d^2y}{dx^2} = 0$ at $x = 1$

$\frac{d^2y}{dx^2} = 6ax + 2b$

$6a(1) + 2b = 0$

$2b = -6a$

$b = -3a$..(iv)

Combining (ii), (iii) and (iv):

$a + b + c = -4$

$a - 3a + 0 = -4$

$-2a = -4$

$a = 2$ $\Rightarrow b = -6$

Hence, $a = 2$: $b = -6$: $c = 0$: $d = 4$ [giving $f(x): 2x^3 - 6x^2 + 4$]

Blunders (–3)

B1 Differentiation.

B2 Indices.

Slips (–1)

S1 Numerical.

Worthless (0)

W1 No $f''(x)$.

Note Equations must come from work above.

QUESTION 7

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**
Part (c)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**

Part (a) **10 marks** **Att 3**

7(a) Find the slope of the tangent to the curve

$9x^2 + 4y^2 = 40$ at the point (2, 1).

Part (a) **10 marks** **Att 3**

7(a)

$$9x^2 + 4y^2 = 40$$

$$18x + 8y\frac{dy}{dx} = 0$$

$$8y\frac{dy}{dx} = -18x$$

$$\frac{dy}{dx} = \frac{-18x}{8y} = \frac{-9x}{4y}$$

At $x = 2$, $y = 1$ $\quad m = \frac{dy}{dx} = \frac{-9(2)}{4(1)} = \frac{-9}{2}$.

Blunders (–3)
B1 Differentiation.
B2 Incorrect value of x or no value of x.
B3 Incorrect value of y or no value of y.
B4 Indices.

Slips (–1)
S1 Numerical.

Attempts
A1 Error in differentiation formula.
A2 $\frac{dy}{dx} = 18x + 8y\frac{dy}{dx}$ and uses the two $\left(\frac{dy}{dx}\right)$ terms.

Worthless (0)
W1 Integration.
W2 No differentiation.

Part (b)(i) **10 marks** **Att 3**

7(b)(i) Given that $y = \sin^{-1} 10x$, evaluate $\dfrac{dy}{dx}$ when $x = \dfrac{1}{20}$.

Part(b)(i) **10 marks** **Att 3**

7(b)(i)

$$y = \sin^{-1}(10x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-100x^2}} \cdot 10 = \frac{10}{\sqrt{1-100x^2}}$$

At $x = \dfrac{1}{20}$:

$$\frac{dy}{dx} = \frac{10}{\sqrt{1-\frac{100}{400}}} = \frac{10}{\sqrt{1-\frac{1}{4}}} = \frac{10}{\sqrt{\frac{3}{4}}}$$

$$\frac{dy}{dx} = \frac{10}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{20}{\sqrt{3}}$$

or

7(b)(i)

$$y = \sin^{-1}(10x)$$

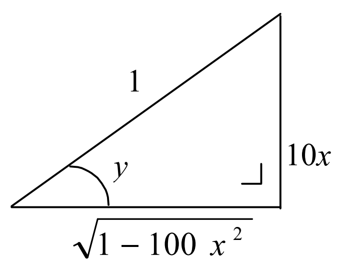
$$\sin y = 10x$$

$$\cos y \frac{dy}{dx} = 10$$

$$\frac{dy}{dx} = \frac{10}{\cos y} = \frac{10}{\sqrt{1-100x^2}} \quad \text{(from diagram)}$$

At $x = \dfrac{1}{20}$:

$$\frac{dy}{dx} = \frac{10}{\sqrt{1-\frac{100}{400}}} = \frac{10}{\sqrt{\frac{3}{4}}} = \frac{20}{\sqrt{3}}$$



$$\sin y = 10x = \frac{10x}{1}$$

$$\cos y = \frac{\sqrt{1-100x^2}}{1} = \sqrt{1-100x^2}$$

Blunders (–3)

B1 Differentiation.
B2 Indices.
B3 Definition of sin y and/or cos y (once only).
B4 Sides of triangle (once only).
B5 Not evaluating derivative at given x.

Attempts

A1 Error in differentiation formula.

Worthless (0)

W1 Integration.

Part (b)(ii) **10 (5, 5)marks** **Att (2, 2)**

7(b)(ii) The parametric equations of a curve are

$$x = ln(1+t^2) \text{ and } y = ln 2t \text{, where } t \in R, t > 0.$$

Find the value of $\frac{dy}{dx}$ when $t = \sqrt{5}$.

$\frac{dx}{dt}, \frac{dy}{dt}$ **5 marks** **Att 2**

Value **5 marks** **Att 2**

7(b)(ii)

$$y = ln 2t \qquad x = ln(1+t^2)$$

$$\frac{dy}{dt} = \frac{1}{2t} \cdot 2 = \frac{1}{t} \qquad \frac{dx}{dt} = \frac{1}{1+t^2} \cdot 2t = \frac{2t}{1+t^2}$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{1}{t}\right)}{\left(\frac{2t}{1+t^2}\right)} = \frac{1+t^2}{2t^2}$$

When $t = \sqrt{5}$, $\frac{dy}{dx} = \frac{1+5}{2(5)} = \frac{3}{5}$

Blunders (–3)

B1 Differentiation.
B2 Indices.
B3 Definition of $\frac{dy}{dx}$.
B4 Logs.
B5 Value or no value.

Attempts

A1 Error in differentiation formula.

Worthless

W1 Integration.
W2 No differentiation.

Part (c)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**

7(c) Let $f(x) = \frac{e^x + e^{-x}}{2}$

(i) Show that $f''(x) = f(x)$, where $f''(x)$ is the second derivative of $f(x)$.

(ii) Show that $\frac{f'(2x)}{f'(x)} = 2f(x)$ when $x \neq 0$ and where $f'(x)$ is the first derivative of *f(x)*

Part (c)(i)	**10 marks**	**Att 3**
$f'(2x)$	**5 marks**	**Att 2**
Finish	**5 marks**	**Att 2**

7(c)(i)

$$f(x) = \frac{1}{2}\left[e^x + e^{-x}\right]$$

$$f'(x) = \frac{1}{2}\left[e^x - e^{-x}\right]$$

$$f''(x) = \frac{1}{2}\left[e^x + e^{-x}\right] \qquad \Rightarrow f(x) = f''(x)$$

7(c)(ii)

$$\frac{f'(2x)}{f'(x)} = 2[f(x)]$$

$$f'(x) = \frac{1}{2}\left[e^x - e^{-x}\right]$$

$$f'(2x) = \frac{1}{2}\left[e^{2x} - e^{-2x}\right] = \frac{1}{2}\left[\left(e^x\right)^2 - \left(e^{-x}\right)^2\right]$$

$$\frac{f'(2x)}{f'(x)} = \frac{\frac{1}{2}\left[\left(e^x\right)^2 - \left(e^{-x}\right)^2\right]}{\frac{1}{2}\left[e^x - e^{-x}\right]} = \frac{\frac{1}{2}\left[\left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right)\right]}{\frac{1}{2}\left[e^x - e^{-x}\right]}$$

$$= e^x + e^{-x}$$

$$= 2\left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]$$

$$= 2[f(x)]$$

Blunders (–3)

B1 Differentiation.

B2 Indices.

B3 $f'(2x)$.

Attempts

A1 Error in differentiation formula.

Worthless (0)

W1 No differentiation.

W2 Integration.

QUESTION 8

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (10, 10)**	**Att (3, 3)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 marks**	**Att 3**

8(a)

Find $\int \left(x^3 + \sqrt{x} + 2\right) dx$.

Part (a)	**10 marks**	**Att 3**

8(a)

$$\int \left(x^3 + x^{\frac{1}{2}} + 2\right) dx = \frac{x^4}{4} + \frac{x^{\frac{1}{2}}}{\frac{3}{2}} + 2x + c$$

$$= \frac{x^4}{4} + \frac{2}{3}x^{\frac{3}{2}} + 2x + c$$

Blunders (–3)
B1 Integration.
B2 No ‘c’.
B3 Indices.

Attempts
A1 Only c correct.

Worthless
W1 Differentiation instead of integration.

Part (b)	20 (10, 10) marks	Att (3, 3)

8(b)

Evaluate **(i)** $\int_2^7 \frac{2x-4}{x^2-4x+29}dx$ **(ii)** $\int_2^7 \frac{1}{x^2-4x+29}dx$.

Part (i)	10 marks	Att 3
Part(ii)	10 marks	Att 3

8(b)(i)

$$\int_2^7 \frac{(2x-4)\,dx}{x^2-4x+29}$$

Let $w = x^2 - 4x + 29 \Rightarrow \frac{dw}{dx} = 2x - 4$

$\Rightarrow dw = (2x-4)dx$

Change limits: $x = 2, u = 25$ and $x = 7, u = 50$

$$\int_2^7 \frac{(2x-4)\,dx}{x^2-4x+29} = \int_{25}^{50} \frac{dw}{w}$$

$$= \ln w \Big|_{25}^{50}$$

$$= \ln 50 - \ln 25$$

$$= \ln\left(\frac{50}{25}\right) = \ln 2$$

or

$$= \ln\left(x^2 - 4x + 29\right) \Big]_2^7$$

$$= \ln[49 - 28 + 29] - \ln[4 - 8 + 29]$$

$$= \ln 50 - \ln 25$$

$$= \ln\left(\frac{50}{25}\right) = \ln 2$$

8(b)(ii)

$$\int_2^7 \frac{dx}{x^2-4x+29}$$

$$x^2 - 4x + 29 = x^2 - 4x + 4 + 25$$

$$= (x-2)^2 + (5)^2$$

$$\int \frac{dx}{x^2-4x+29} = \int \frac{dx}{(x-2)^2+(5)^2} \qquad w = x - 2$$

$$= \int \frac{dw}{w^2+5^2} \qquad \frac{dw}{dx} = 1$$

$$dw = dx$$

$$= \frac{1}{5}\tan^{-1}\left(\frac{w}{5}\right)$$

$$\int_2^7 \frac{dx}{x^2 - 4x + 29} = \frac{1}{5}\left[\tan^{-1}\left(\frac{x-2}{5}\right)\right]_2^7 \quad \textbf{or} \quad \frac{1}{5}\tan^{-1}\left(\frac{w}{5}\right)\Bigg|_0^5$$

$$= \frac{1}{5}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1}{5}\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{20}$$

Blunders (–3)
B1 Integration.
B2 Indices.
B3 Limits.
B4 No limits.
B5 Incorrect order in applying limits.
B6 Not calculating substituted limits.
B7 Not changing limits.
B8 Differentiation.

Slips (–1)
S1 Numerical.
S2 Uses $\pi = 180^\circ$.

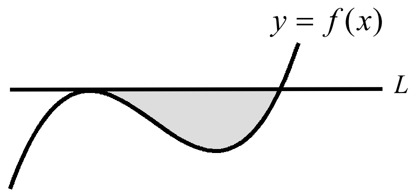
Worthless:
W1 Differentiation instead of integration (except where other work merits attempt).
W2 Puts $w = x^2 - 4x + 29$ in **(ii)**.

Note Incorrect substitution and unable to finish merits attempt at most.
Note (–3) is maximum deduction in evaluation of limits.

Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

8(c) Let $f(x) = x^3 - 3x^2 + 5$.
L is the tangent to the curve $y = f(x)$ at its local maximum point

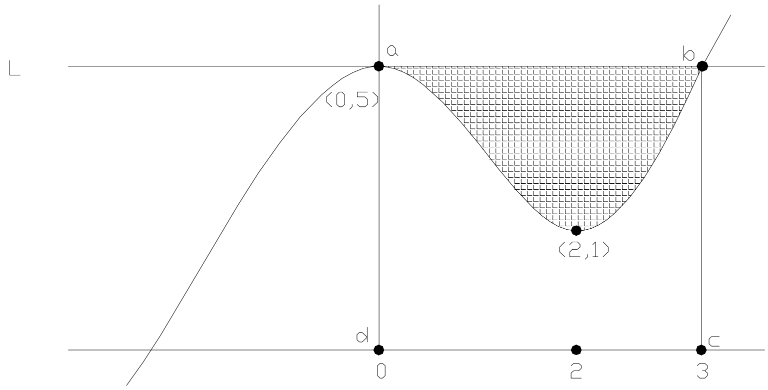
Find the area enclosed between L and the curve.



Max Point	**5 marks**	**Att 2**
Other Limit	**5 marks**	**Att 2**
1st Area	**5 marks**	**Att 2**
2nd Area	**5 marks**	**Att 2**

8(c)

$$f(x) = x^3 - 3x^2 + 5$$
$$f'(x) = 3x^2 - 6x$$
$$f''(x) = 6x - 6$$



Local max/min at $f'(x) = 0$

$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$
$$x = 0 \quad \text{or} \quad x = 2$$

Test in $f''(x) = 6x - 6$

$x = 0$: $6(0) - 6 = -6 < 0 \Rightarrow$ Local max at $x = 0$

$x = 0$: $f(x) = y = 0 - 0 + 5 = 5 \Rightarrow$ Local max at (0, 5)

Equation L: $y = 5$

L $\cap$ Curve: $\{y = 5\} \cap \{y = x^3 - 3x^2 + 5\}$

$$5 = x^3 - 3x^2 + 5$$
$$0 = x^2(x-3)$$
$$x = 0 \quad \text{or} \quad x = 3$$

$x = 0,\ y = 5 \Rightarrow a(0,5)$ $\qquad$ $x = 3,\ y = 5 \Rightarrow b(3,5)$

Required area = Area rectangle $abcd - \int_0^3 y\,dx$

Area $abcd$ = (5)(3) = 15 **or** area rectangle = $\int_0^3 y \cdot dx = \int 5 \cdot dx = 5x \Big]_0^3 = 15 - 0 = 15$

$$\int_0^3 y \cdot dx = \int (x^3 - 3x^2 + 5)dx = \left(\frac{x^4}{4} - x^3 + 5x\right)_0^3$$
$$= \left(\frac{81}{4} - 27 + 15\right) - 0$$
$$= \frac{33}{4}$$

Required Area = $15 - \frac{33}{4} = \frac{60 - 33}{4} = \frac{27}{4}$

Blunders (–3)

B1 Indices.
B2 Integration.
B3 $f'(x) \neq 0$.
B4 Factors (once only).
B5 Value from factor.
B6 Area formula.
B7 Limits.
B8 Not changing limits.
B9 Not finding required area.

Attempts

A1 Some relevant area.
A2 Uses volume formula.
A3 Uses y^2 in formula.

Slips (–1)

S1 Numerical

Worthless (0)

W1 Differentiation instead of integration except where other work merits attempts.
W2 Wrong area formula and no work.
W3 Graphical methods only.

Notes

N1 (–3) is maximum deduction when evaluating limits.
N2 Where candidate obtains area by translation, allow 5 marks for translation and 5 marks for required area.

An Roinn Oideachais agus Eolaíochta

Leaving Certificate Examination 2002

Marking Scheme

MATHEMATICS

Higher Level

Paper 2

General Instructions

Penalties are applied as follows:

numerical slips, misreadings	(−1) each
blunders, major omissions	(−3) each.

Note 1: The lists of slips, blunders and attempts given in the marking scheme are not exhaustive.

Note 2: A serious blunder, omission or misreading merits the attempt mark at most.

Note 3: The attempt mark for a section is the final mark for that section. Where deductions result in a mark that is less than the attempt mark, the attempt mark is awarded.

Note 4: Particular cases and verifications are, in general, awarded the attempt mark only.

Note 5: All of the candidate's work, including any that is cancelled, is marked and the highest scoring solutions are allowed.

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QUESTION 1

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

1(a)

The following parametric equations define a circle:

$$x = 4 + 3\cos\theta, \quad y = -2 + 3\sin\theta, \quad \text{where } \theta \in \mathbf{R}.$$

What is the Cartesian equation of the circle?

Circle Equation **10 marks** **Att 3**

1(a)

$$x = 4 + 3\cos\theta, \quad y = -2 + 3\sin\theta, \quad \text{where } \theta \in \mathbf{R}.$$

$$(x-4)^2 = 9\cos^2\theta, \quad (y+2)^2 = 9\sin^2\theta.$$

$$(x-4)^2 + (y+2)^2 = 9(\cos^2\theta + \sin^2\theta)$$

$$\therefore \quad (x-4)^2 + (y+2)^2 = 9.$$

Blunders (–3)

B1 Incorrect squaring.

B2 Incorrect centre or incorrect radius.

Slips (–1)

S1 Arithmetic error.

Attempts (3 marks)

A1 $x - 4 = 3\cos\theta$ or $y + 2 = 3\sin\theta$.

A2 Correct centre or correct radius.

A3 $(x-4)^2 + (y+2)^2 = 9\cos^2\theta + 9\sin^2\theta$ and stops.

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

Part (b)(i) **5 marks** **Att 2**

1(b)(i)

The points $a(-2, 4)$, $b(0, -10)$ and $c(6, -2)$ are the vertices of a triangle.

Verify that the triangle is right-angled at c.

Verify $|\angle acb| = 90^0$ **5 marks** **Att 2**

1(b)(i)

$$\text{slope } ac = \frac{-2-4}{6+2} = \frac{-6}{8} = \frac{-3}{4} = m_1$$

$$\text{slope } bc = \frac{-2+10}{6-0} = \frac{8}{6} = \frac{4}{3} = m_2$$

But $m_1.m_2 = -1 \Rightarrow ac \perp bc$.

or

$$|ab|^2 = (0+2)^2 + (-10-4)^2 = 200.$$

$$|bc|^2 = (6-0)^2 + (-2+10)^2 = 100.$$

$$|ac|^2 = (6+2)^2 + (-2-4)^2 = 100.$$

As $|ab|^2 = |bc|^2 + |ac|^2 \Rightarrow ac \perp bc$.

Blunders (–3)
B1 Error in slope formula.
B2 Error in distance formula.
B3 Incorrect application of Pythagoras.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 One slope found.
A2 Length of one line found.
A3 $m_1.m_2$ not stated.

Part (b) (ii) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

1(b)(ii)	Hence, or otherwise, find the equation of the circle that passes through the points a, b and c.

Centre or three equations in *g*, *f*, *c*	**5 marks**	**Att 2**
Radius or *g*, *f*, *c* solved	**5 marks**	**Att 2**
Equation of circle	**5 marks**	**Att 2**

1(b)(ii)

$a(-2, 4)$, $b(0, -10)$. $[ab]$ is diameter (verified in **(b)(i)**).

$\therefore$ centre of circle is midpoint of $[ab]$.

Centre is $d\left(\frac{-2-0}{2}, \frac{4-10}{2}\right) = d(-1, -3)$.

Radius $= |ad| = \sqrt{(-2+1)^2 + (4+3)^2} = \sqrt{50}$.

$\therefore$ Circle: $(x+1)^2 + (y+3)^2 = 50$.

or

$C: x^2 + y^2 + 2gx + 2fy + c = 0$.

$a(-2, 4) \in C \Rightarrow -4g + 8f + c = -20$.

$b(0, -10) \in C \Rightarrow -20f + c = -100$.

$c(6, -2) \in C \Rightarrow 12g - 4f + c = -40$.

$$\begin{array}{r} -12g + 24f + 3c = -60 \\ 12g - 4f + c = -40 \\ \hline 20f + 4c = -100 \end{array} \Rightarrow 5f + c = -25.$$

$$\begin{array}{r} 5f + c = -25 \\ -20f + c = -100 \\ \hline 25f = 75 \end{array} \Rightarrow f = 3.$$

Substituting gives $c = -40$ and $g = 1$.

$\therefore$ $C: x^2 + y^2 + 2x + 6y - 40 = 0$.

Blunders (–3)

B1 Error in midpoint formula.

B2 Incorrect centre, e.g. not taking $[ab]$ as diameter.

B3 Incorrect distance formula.

B4 Diameter for radius.

B5 Incorrect form of circle equation.

Slips (–1)

S1 Arithmetic error.

Attempts (2 marks)

A1 $[ab]$ diameter.

A2 One equation in g, f and c.

A3 Equation of circle with some substitution.

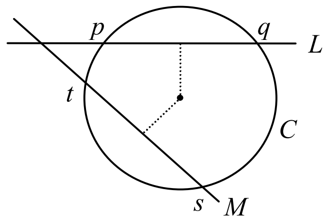
Part (c) 20 (5, 5, 5, 5) marks Att (2, 2, 2, 2)

Part (c)(i) 5 marks Att 2

1(c)(i)

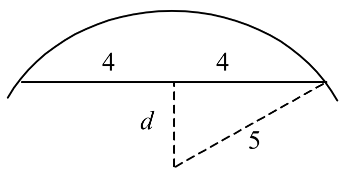
The circle C has equation $x^2 + y^2 - 4x + 6y - 12 = 0$.
L intersects C at the points p and q.
M intersects C at the points t and s.
$|pq| = |ts| = 8$.

Find the radius of C and hence show that the distance from the centre of C to each of the lines L and M is 3.



Show distance from centre = 3 **5 marks** **Att 2**

1(c)(i)



$x^2 + y^2 - 4x + 6y - 12 = 0$ Centre is $(2,-3)$

$r = \sqrt{g^2 + f^2 - c} = \sqrt{4 + 9 + 12} = 5.$

$5^2 = d^2 + 4^2$

$\therefore \; d = 3.$

Blunders (–3)
B1 Incorrect sign in centre point or incorrect g, f.
B2 Error in radius length formula.
B3 Incorrect application of Pythagoras.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Correct centre.
A2 Correct radius.
A3 ½| length of chord| = 4.

Part (c)(ii)	**15 (5, 5, 5) marks**	**Att (2, 2, 2)**

1(c)(ii)	Given that L and M intersect at the point (–4, 0), find the equations of L and M.

Equation of line with full substitution	**5 marks**	**Att 2**
Quadratic $3m^2 + 4m = 0$	**5 marks**	**Att 2**
To finish	**5 marks**	**Att 2**

1(c)(ii)

Required equations: $y - 0 = m(x + 4)$

$$mx - y + 4m = 0$$

But perpendicular distance from centre (2, –3) to $mx - y + 4m = 0$ equals 3.

$$\therefore \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right| = 3 \Rightarrow \left| \frac{2m + 3 + 4m}{\sqrt{m^2 + 1}} \right| = 3.$$

$$6m + 3 = 3\sqrt{m^2 + 1} \Rightarrow (2m + 1)^2 = m^2 + 1.$$

$$3m^2 + 4m = 0 \Rightarrow m(3m + 4) = 0.$$

$$\therefore m = 0 \text{ or } m = -\frac{4}{3}.$$

$mx - y + 4m = 0$, when $m = 0 \Rightarrow L: y = 0.$

$mx - y + 4m = 0$, when $m = -\frac{4}{3} \Rightarrow M: 4x + 3y + 16 = 0.$

Blunders (–3)
B1 Error in line formula.
B2 Error in ⊥ distance formula.
B3 Incorrect squaring.
B4 Incorrect factors.
B5 m found but lines not given.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Equation of line with some substitution.
A2 Substitution into ⊥ distance formula.
A3 One value of m.

QUESTION 2

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 10, 5) marks**	**Att (2, 3, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a) **10 marks** **Att 3**

2(a) $\vec{s} = 4\vec{i} - 3\vec{j}$ and $\vec{t} = 2\vec{i} - 5\vec{j}$.

Find $|\overrightarrow{st}|$.

$|\overrightarrow{st}|$ **10 marks** **Att 3**

2(a)

$\vec{s} = 4\vec{i} - 3\vec{j}$ and $\vec{t} = 2\vec{i} - 5\vec{j}$.

$\overrightarrow{st} = \vec{t} - \vec{s} = 2\vec{i} - 5\vec{j} - 4\vec{i} + 3\vec{j} = -2\vec{i} - 2\vec{j}.$

$\therefore \quad |\overrightarrow{st}| = \sqrt{4+4} = \sqrt{8}.$

Blunders (–3)
B1 Incorrect formula for norm of vector.
B2 Error in distance formula.

Slips (–1)
S1 Arithmetic error.

Attempts (3 marks)
A1 $\overrightarrow{st} = \vec{t} - \vec{s}$ or $\overrightarrow{ts} = \vec{s} - \vec{t}$.

Worthless (0)
W1 $\overrightarrow{st} = \vec{s}.\vec{t}$.

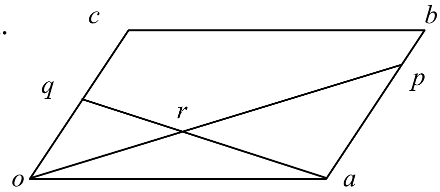
Part (b) 20 (5, 10, 5) marks Att (2, 3, 2)

Part (b)(i) 5 marks Att 2

2(b)(i)

oabc is a parallelogram, where *o* is the origin.
$p \in [ab]$ such that $|ap|:|pb| = 3:1$.
q is the midpoint of [*oc*].

Using equiangular triangles, or otherwise, find the ratio $|or|:|rp|$.



Find ratio $|or|:|rp|$ 5 marks Att 2

2(b)(i)

$$\frac{|or|}{|rp|} = \frac{|oq|}{|ap|} = \frac{\frac{1}{2}|oc|}{\frac{3}{4}|ab|} = \frac{2|oc|}{3|oc|} = \frac{2}{3}.$$

Blunders (–3)
B1 Error in ratio of sides.
B2 $|oq|$ or $|ap|$ incorrect in relation to $|oc|$ or $|ab|$ respectively.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Correct equiangular triangles.
A2 $|oq|$ correct in relation to $|oc|$.
A3 $|ap|$ correct in relation to $|ab|$.

Part (b)(ii)	**15 (10, 5) marks**	**Att (3, 2)**

2(b)(ii) Express $\vec{p}$, and hence $\vec{r}$, in terms of $\vec{a}$ and $\vec{b}$.

Express $\vec{p}$	**10 marks**	**Att 3**
Express $\vec{r}$	**5 marks**	**Att 2**

2(b)(ii)

$$\vec{p} = \frac{\vec{a}+3\vec{b}}{4} = \tfrac{1}{4}\vec{a} + \tfrac{3}{4}\vec{b}$$

or

$$\vec{p} = \vec{a} + \tfrac{3}{4}\overrightarrow{ab} = \vec{a} + \tfrac{3}{4}(\vec{b}-\vec{a})$$
$$\vec{p} = \tfrac{1}{4}\vec{a} + \tfrac{3}{4}\vec{b}$$

$$\vec{r} = \frac{3\vec{o}+2\vec{p}}{5} = \tfrac{2}{5}\vec{p} = \tfrac{2}{5}(\tfrac{1}{4}a + \tfrac{3}{4}\vec{b})$$
$$= \tfrac{1}{10}\vec{a} + \tfrac{3}{10}\vec{b}$$

or

$$|or| : |rp| = 2 : 3 \Rightarrow \vec{r} = \tfrac{2}{5}\vec{p} = \tfrac{2}{5}(\tfrac{1}{4}\vec{a} + \tfrac{3}{4}\vec{b})$$
$$\vec{r} = \tfrac{1}{10}\vec{a} + \tfrac{3}{10}\vec{b}$$

Blunders (–3)

B1 Error in ratio formula.

B2 $\overrightarrow{ab} = \vec{a} - \vec{b}$.

B3 $r\vec{p} = \vec{r} - \vec{p}$.

B4 Incorrect value of $|or| : |rp|$.

Slips (–1)

S1 Arithmetic error.

Attempts (3, 2 marks)

A1 $\vec{p} = \vec{a} + \tfrac{3}{4}\overrightarrow{ab}$.

A2 $\vec{r} = \tfrac{2}{5}\vec{p}$.

A3 $3\vec{r} = 2\overrightarrow{rp}$.

Part (c)	20 (5, 5, 5, 5) marks	Att (2, 2, 2, 2)

Part (c)(i)	10 marks (5, 5)	Att (2, 2)

2(c)(i) $\vec{k} = \vec{i} + 3\vec{j}$, $\vec{n} = 4\vec{i} - 2\vec{j}$, $\vec{u} = 2\vec{i} + \vec{j}$ and $\vec{v} = x\vec{i} + y\vec{j}$ where $x, y \in \mathbf{R}$.

Express the value of $\overrightarrow{kn}.\overrightarrow{kv}$ in the form $ax + by + c$ where $a, b, c \in \mathbf{R}$.

$\overrightarrow{kv}$	**5 marks**	**Att 2**
To finish	**5 marks**	**Att 2**

2(c)(i)

$$\overrightarrow{kn} = \vec{n} - \vec{k} = 4\vec{i} - 2\vec{j} - \vec{i} - 3\vec{j} \Rightarrow \overrightarrow{kn} = 3\vec{i} - 5\vec{j}.$$

$$\overrightarrow{kv} = \vec{v} - \vec{k} = x\vec{i} + y\vec{j} - \vec{i} - 3\vec{j} \Rightarrow \overrightarrow{kv} = (x-1)\vec{i} + (y-3)\vec{j}.$$

$$\overrightarrow{kn}.\overrightarrow{kv} = (3\vec{i} - 5\vec{j}).\left((x-1)\vec{i} + (y-3)\vec{j}\right)$$

$$\overrightarrow{kn}.\overrightarrow{kv} = 3(x-1) - 5(y-3) = 3x - 5y + 12.$$

Blunders (–3)

B1 $kv = \overrightarrow{k} - \vec{v}$.

B2 Incorrect vector multiplication.

Slips (–1)

S1 Arithmetic error.

Attempts (2 marks)

A1 $\overrightarrow{kv} = \vec{v} - \vec{k}$.

A2 $\overrightarrow{kn}$ correct.

Part (c)(ii) **10 (5, 5) marks** **Att (2, 2)**

2(c)(ii)	Prove that if $\overrightarrow{kn}.\overrightarrow{kv} = \overrightarrow{kn}.\overrightarrow{ku}$, and $\vec{u} \neq \vec{v}$, then $\overrightarrow{kn} \perp \overrightarrow{uv}$.

$\overrightarrow{kn}.\overrightarrow{ku}$	**5 marks**	**Att 2**
To finish	**5 marks**	**Att 2**

2(c)(ii)

$$\overrightarrow{ku} = \vec{u} - \vec{k} = 2\vec{i} + \vec{j} - \vec{i} - 3\vec{j} \Rightarrow \overrightarrow{ku} = \vec{i} - 2\vec{j}.$$

But $\overrightarrow{kn} = 3\vec{i} - 5\vec{j} \Rightarrow \overrightarrow{kn}.\overrightarrow{ku} = (3\vec{i} - 5\vec{j}).(\vec{i} - 2\vec{j}) = 3 + 10 = 13.$

From part **(i)** $\overrightarrow{kn}.\overrightarrow{kv} = 3x - 5y + 12.$

$\therefore 3x - 5y + 12 = 13 \Rightarrow 3x - 5y - 1 = 0$ when $\overrightarrow{kn}.\overrightarrow{kv} = \overrightarrow{kn}.\overrightarrow{ku}.$

$$\overrightarrow{uv} = \vec{v} - \vec{u} = x\vec{i} + y\vec{j} - 2\vec{i} - \vec{j} \Rightarrow \overrightarrow{uv} = (x-2)\vec{i} + (y-1)\vec{j}.$$

$$\begin{aligned}\overrightarrow{kn}.\overrightarrow{uv} &= (3\vec{i} - 5\vec{j}).\big((x-2)\vec{i} + (y-1)\vec{j}\big) \\ &= 3(x-2) - 5(y-1) = 3x - 5y - 1 \\ &= 0\end{aligned}$$

$$\therefore \overrightarrow{kn} \perp \overrightarrow{uv}.$$

Blunders (–3)

B1 Error in finding scalar $\overrightarrow{kn}.\overrightarrow{ku}$.

B2 No conclusion given why $\overrightarrow{kn} \perp \overrightarrow{uv}$.

Slips (–1)

S1 Arithmetic error.

Attempts (2 marks)

A1 $\overrightarrow{ku}$ correct.

A2 $\overrightarrow{uv}$ correct.

QUESTION 3

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**40 (5, 10, 10, 5, 5, 5) marks**	**Att (2, 3, 3, 2, 2,2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

3(a) $a(-1, 4)$ and $b(5, -4)$ are two points.

Find the equation of the perpendicular bisector of $[ab]$.

Midpoint or slope *ab*	**5 marks**	**Att 2**
To finish	**5 marks**	**Att 2**

3(a)

$$a(-1, 4),\ b(5, -4) \ \Rightarrow\ \text{midpoint } [ab] = (2, 0)$$

$$\text{slope } ab = \frac{-4-4}{5+1} = \frac{-8}{6} = -\frac{4}{3} \ \Rightarrow\ \perp \text{ slope } = \frac{3}{4}$$

$$\text{Equation of perpendicular bisector : } y - 0 = \frac{3}{4}(x-2)$$

$$3x - 4y - 6 = 0.$$

Blunders (–3)
B1 Error in midpoint formula.
B2 Error in slope formula.
B3 Incorrect $\perp$ slope.
B4 Error in equation of line formula.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Midpoint with some substitution.
A2 Slope formula with some substitution.
A3 Equation of line *ab*.
A4 Line formula with some substitution.

Part (b)	40 (5, 10, 10, 5, 5, 5) marks	Att (2, 3, 3, 2, 2,2)

Part (b)(i) **5 marks** **Att 2**

3(b)(i)

f is the transformation $(x, y) \to (x', y')$ where $x' = 3x + y$ and $y' = x - 2y$.
S is the square whose vertices are (0, 0), (1, 0), (1, 1) and (0, 1).

(i) Find the image under f of each of the four vertices of S.

Image points **5 marks** **Att 2**

3(b)(i)

$f(0, 0) = (0, 0)$	**or**	$\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$f(1, 0) = (3, 1)$		$\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
$f(1, 1) = (4, -1)$		$\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$
$f(0, 1) = (1, -2)$		$\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Blunders (–3)
B1 Image point incorrect (unless due to slip).
B2 Incorrect matrix multiplication (unless due to slip).

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 One correct image point.
A2 Correct matrix for f.

Part (b)(ii)	10 marks	Att 3

3(b)(ii) Express x and y in terms of x' and y'.

Express x and y	10 marks	Att 3

3(b)(ii)

f is the transformation $(x, y) \to (x', y')$ where $x' = 3x + y$ and $y' = x - 2y$.

$$2x' = 6x + 2y$$
$$y' = x - 2y$$
$$2x' + y' = 7x \Rightarrow x = \tfrac{1}{7}(2x' + y')$$

But $y = x' - 3x \Rightarrow y = x' - \tfrac{3}{7}(2x' + y')$

$\therefore \; y = \tfrac{1}{7}(x' - 3y')$.

or

$$\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{-1}{7}\begin{pmatrix} -2 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$\Rightarrow \; x = \tfrac{1}{7}(2x' + y')$ and $y = \tfrac{1}{7}(x' - 3y')$.

Blunders (–3)
B1 Error in matrix f.
B2 Error in f^{-1}.

Slips (–1)
S1 Arithmetic error.

Attempts (3 marks)
A1 Attempt at expressing x or y in terms of primes.
A2 Correct matrix for f.

Part (b)(iii) **10 marks** **Att 3**

3(b)(iii)

By considering the lines $ax+by+c=0$ and $ax+by+d=0$, or otherwise, prove that *f* maps every pair of parallel lines to a pair of parallel lines.
(You may assume that *f* maps every line to a line.)

Prove **10 marks** **Att 3**

3(b)(iii)

L: $ax+by+c=0$

$f(L)$: $f(ax+by+c)=0$

$f(L)$: $\frac{a}{7}(2x'+y')+\frac{b}{7}(x'-3y')+c = 0$

$f(L)$: $x'(2a+b)+y'(a-3b)+7c = 0$

Similarly, M: $ax+by+d=0$

$f(M)$: $f(ax+by+d)=0$

$f(M)$: $x'(2a+b)+y'(a-3b)+7d = 0$

$$\text{Slope } f(L) = \frac{-(2a+b)}{a-3b} = \text{Slope } f(M)$$

$\therefore$ L parallel to M $\Rightarrow$ $f(L)$ parallel to $f(M)$.

Blunders (–3)

B1 Incorrect matrix or matrix multiplication.

B2 Image line not simplified to $px'+qy'+r=0$.

B3 Failure to finish correctly.

Slips (–1)

S1 Arithmetic error.

Attempts (3 marks)

A1 Correct substitution of primes.

A2 Correct matrix for *f*.

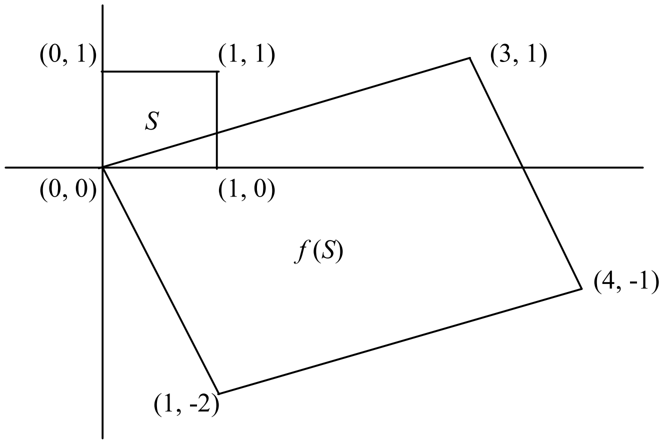
A3 Finds image of one line and stops.

Part (b)(iv)	**10 marks (5, 5)**	**Att (2, 2)**

3(b)(iv) Show both S and $f(S)$ on a diagram.

Show *S*	**5 marks**	**Att 2**
Show *f* (*S*)	**5 marks**	**Att 2**

3(b)(iv)



Blunders (–3)
B1 Incorrect plotting of point for *S*.
B2 Incorrect plotting of point for $f(S)$.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Diagram showing two points for *S*.
A2 Diagram showing two points for $f(S)$.

Part (b)(v)	**5 marks**	**Att 2**

3(b)(v) Find the area of $f(S)$.

Area of *f* (*S*)	**5 marks**	**Att 2**

3(b)(v) Area $f(S)$ = 2 × area of triangle with vertices (0, 0), (4, –1), (3, 1).

$= 2\times\frac{1}{2}\left|4(1) - 3(-1)\right| = 7$ square units.

or Matrix $f = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix} \Rightarrow \left|\det f\right| = 7.$

But area $f(S) = \left|\det f\right| \times$ area $S = 7\times1 = 7$ square units.

Blunders (–3)
B1 Error in area of triangle formula.
B2 Takes $f(S)$ as rectangle.
B3 Error in determinant of *f*.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Area of a triangle found.
A2 Correct determinant.

QUESTION 4

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (10, 5, 5)marks**	**Att (3, 2, 2)**

Part (a)	**10 marks**	**Att 3**

4(a)

Find the value of θ for which $\cos\theta = -\frac{\sqrt{3}}{2}$, $0° \le \theta \le 180°$.

Find the value of θ	**10 marks**	**Att 3**

4(a)

$$\cos\theta = -\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = 150^0.$$

Blunders (–3)
B1 Solution = 30^0, 150^0.

Slips (–1)
S1 Arithmetic error.

Attempts (3 marks)
A1 Solution = 30^0.
A2 Solution = 30^0, 120^0.

Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (b)(i)	**5 marks**	**Att 2**

4(b)(i)

Use the formula $\sin^2 A = \frac{1}{2}(1-\cos 2A)$ to express $\sin^2 \frac{1}{2}x$ in terms of $\cos x$.

Express	**5 marks**	**Att 2**

4(b)(i)

$$\sin^2 \frac{1}{2}x = \frac{1}{2}(1-\cos x).$$

Blunders (–3)
B1 $\sin^2 \frac{1}{2}x = \frac{1}{2}(1-\cos 2x)$.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 $A = \frac{1}{2}x$.

Part (b)(ii) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

4(b)(ii)	Hence, or otherwise, find all the solutions of the equation $$\sin^2 \frac{1}{2}x - \cos^2 x = 0 \quad \text{in the domain } 0° \le x \le 360°.$$

Quadratic in cos*x*	**5 marks**	**Att 2**
Solve for cos*x*	**5 marks**	**Att 2**
Solve for *x*	**5 marks**	**Att 2**

4(b)(ii)

$$\sin^2 \frac{1}{2}x - \cos^2 x = 0$$

$$\tfrac{1}{2}(1 - \cos x) - \cos^2 x = 0$$

$$1 - \cos x - 2\cos^2 x = 0 \quad \Rightarrow \quad 2\cos^2 x + \cos x - 1 = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0$$

$$\therefore \cos x = \frac{1}{2} \quad \text{or} \quad \cos x = -1$$

$$x = 60^0,\ 300^0 \quad \text{or} \quad x = 180^0.$$

$$\text{Solution} = \{60^0,\ 180^0,\ 300^0\}.$$

Blunders (–3)
B1 Error in factors.
B2 Error in quadratic formula.
B3 Missing solution or incorrect 'solution'.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 $\sin^2 \frac{1}{2}x$ replaced by $\frac{1}{2}(1 - \cos x)$.
A2 Correct factors.
A3 One correct solution for x.

Part (c) **20 (10, 5, 5)marks** **Att (3, 2, 2)**

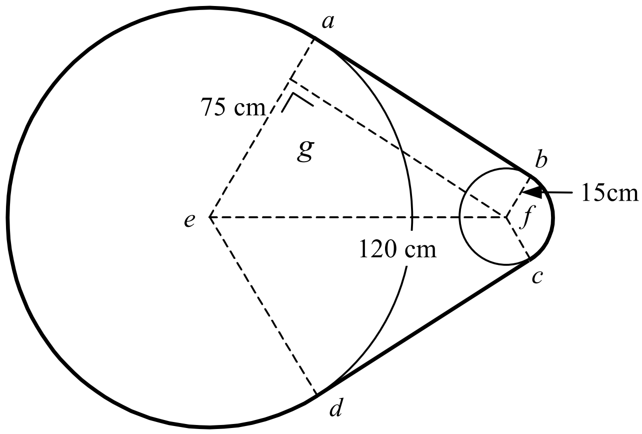
Part (c)(i) **10 marks** **Att 3**

4(c)(i)

A chain passes around two circular wheels as shown. One wheel has radius 75 cm and the other has radius 15 cm. The centres, e and f, of the wheels are 120 cm apart.

The chain consists of the common tangent $[ab]$, the minor arc bc, the common tangent $[cd]$ and the major arc da.

Find the measure of $\angle aef$.



Find the measure of $\angle aef$. **10 marks** **Att 3**

4(c(i)

$$|eg| = 75 - 15 = 60 \text{ cm.}$$

$$\cos\angle aef = \frac{|eg|}{|ef|} = \frac{60}{120} = \frac{1}{2}$$

$$\therefore \ |\angle aef| = 60^0.$$

Blunders (−3)

B1 $\cos^{-1}(½)$ incorrect.

B2 Incorrect ratio of sides for cos.

Slips (−1)

S1 Arithmetic error.

Attempts (3 marks)

A1 $|eg| = 60$ cm.

A2 $\cos\angle aef$ used.

Misreading (−1)

M1 $|eg| = 75$ cm.

Part (c)(ii) **5 marks** **Att 2**

4(c)(ii)	Find $|ab|$ in surd form

Find $|ab|$ **5marks** **Att 2**

4(c)(ii)

$$|ab| = |gf|$$
$$|gf|^2 = |ef|^2 - |eg|^2$$
$$|gf|^2 = 120^2 - 60^2 = 10800.$$
$$|gf| = \sqrt{10800} = 60\sqrt{3}$$
$$\therefore |ab| = 60\sqrt{3}$$

or

$$|eg| = 120\sin 60° = 60\sqrt{3}.$$

Blunders (−3)

B1 sin60^0 not expressed as $\frac{\sqrt{3}}{2}$.

B2 Solution in decimal form.

Slips (−1)

S1 Arithmetic error.

Attempts (2 marks)

A1 $\sin\angle aef$ used.

Part (c)(iii) **5 marks** **Att 2**

4(c)(iii)

Find the length of the chain, giving your answer in the form $k\pi + l\sqrt{3}$ where $k, l \in \mathbf{Z}$.

Find the length of the chain **5 marks** **Att 2**

4(c)(iii)

Length of chain $= |\text{major arc } ad| + |\text{minor arc } bc| + |ab| + |cd|$.

$|\text{major arc } ad| = r\theta$, where $\theta = 240^0 = \dfrac{4\pi}{3}$

$\therefore \; |\text{major arc } ad| = 75\left(\dfrac{4\pi}{3}\right) = 100\pi$

$|\text{minor arc } bc| = r\theta$, where $\theta = 120^0 = \dfrac{2\pi}{3}$

$\therefore \; |\text{minor arc } bc| = 15\left(\dfrac{2\pi}{3}\right) = 10\pi$

Length of the chain $= 100\pi + 10\pi + 60\sqrt{3} + 60\sqrt{3}$

$= 110\pi + 120\sqrt{3}$

Blunders (–3)

B1 Incorrect formula for length of arc.
B2 Incorrect value for θ.
B3 Error in converting from degrees to radians.
B3 Incorrect arc chosen.
B4 Incorrect radius used.
B5 Any length not added for final solution.

Slips (–1)

S1 Arithmetic error.

Attempts (2 marks)

A1 Correct θ given.
A2 Finds 100π or 10π and stops.

Worthless (0)

W1 'Length of arc' given in terms of degrees.

QUESTION 5

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**
Part (c)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**

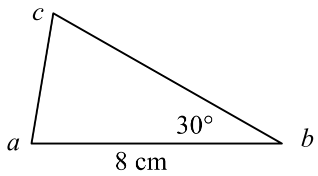
Part (a)	**10 marks**	**Att 3**

5(a)

The area of triangle *abc* is 12 cm^2.

$|ab| = 8$ cm and $|\angle abc| = 30°$.

Find $|bc|$.



Find $|bc|$	**10 marks**	**Att 3**

5(a)

Area triangle *abc* = 12 cm^2

$\frac{1}{2}(8)|bc|\sin 30^0 = 12 \text{ cm}^2$

$2|bc| = 12 \text{ cm} \Rightarrow |bc| = 6 \text{ cm}.$

Blunders (–3)
B1 Error in area of triangle formula.
B2 Sin 30^0 incorrect.

Slips (–1)
S1 Arithmetic error.

Attempts (3 marks)
A1 Area of triangle formula with some substitution.

Worthless (0)
W1 $\angle cab = 90^0$.

Part (b)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**
Part (b)(i)	**15 (10, 5) marks**	**Att (3, 2)**

5(b)(i)

Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

Tan(*A*+*B*) or R.H.S. in terms of sin*A*,cos*B* etc	**10 marks**	**Att 3**
To finish	**5 marks**	**Att 2**

5(b)(i)

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A}.\dfrac{\sin B}{\cos B}} \quad \text{(from dividing above and below by } \cos A\cos B\text{)}$$

$$= \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Blunders (–3)
B1 Incorrect expansion for sin(*A*+*B*) or cos(*A*+*B*).

Slips (–1)
S1 Arithmetic error.

Attempts (3, 2 marks)
A1 Correct expansion for sin(*A*+*B*) or cos(*A*+*B*).
A2 $\text{Tan} = \dfrac{\sin}{\cos}$.
A3 Cannot finish due to error.

Part (b)(ii) **5 marks** **Att 2**

5(b)(ii)	Hence, or otherwise, prove that $\tan 22\tfrac{1}{2}° = \sqrt{2} - 1$.

Prove tan 22½° = $\sqrt{2}$ – 1 **5 marks** **Att 2**

5(b)(ii)

Let $A = B = 22½°$ **or** $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$, where $A = 22\tfrac{1}{2}°$

$$\tan 45° = \frac{2\tan A}{1-\tan^2 A} \Rightarrow 1-\tan^2 A = 2\tan A$$

$$\tan^2 A + 2\tan A - 1 = 0$$

$$\tan A = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

But $A = 22\frac{1}{2}°$ so $\tan A > 0 \Rightarrow = -1 + \sqrt{2}$ is the required value

$$\therefore \tan 22\tfrac{1}{2}° = \sqrt{2} - 1.$$

Blunders (–3)
B1 Error in formula used.
B2 Error in quadratic formula.
B3 Chooses incorrect solution or both solutions.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 $A = B = 22½°$
A2 Use of tan(A+B) expansion.
A3 Use of tan2A.
A4 Fails to solve quadratic.

Worthless (0)
W1 Use of calculator for tan22½°.

Part (c) **20 (10, 5, 5) marks** **Att (3, 2, 2)**

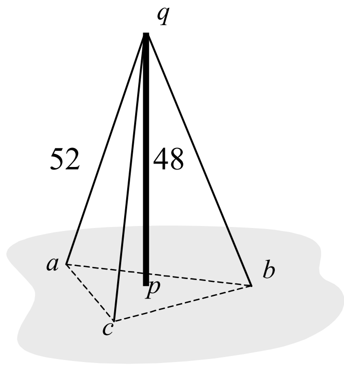
Part (c)(i) **10 marks** **Att 3**

5(c)(i)

A vertical radio mast $[pq]$ stands on flat horizontal ground. It is supported by three cables that join the top of the mast, q, to the points a, b and c on the ground. The foot of the mast, p, lies inside the triangle abc.

Each cable is 52 m long and the mast is 48 m high.

Find the (common) distance from p to each of the points a, b and c.



Find (common) distance from *p* **10 marks** **Att 3**

5(c)(i)

$$|pa| = |pb| = |pc|.$$

triangle apq is right angled at p.

$$\therefore\ |ap|^2 = |aq|^2 - |qp|^2$$

$$|ap|^2 = 52^2 - 48^2 = 400$$

$$|ap| = 20 \text{ metres.}$$

Blunders (–3)

B1 Incorrect squaring.
B2 Incorrect application of Pythagoras.
B3 Error in square root.

Slips (–1)

S1 Arithmetic error

Attempts (3 marks)

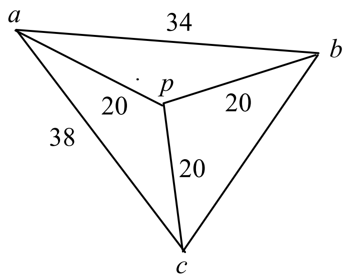
A1 Application of Pythagoras.

Part (c)(ii)	**10 (5, 5) marks**	**Att (2, 2)**

5(c)(ii) Given that $|ac| = 38$ m and $|ab| = 34$ m, find $|bc|$ correct to one decimal place.

Angle from Δpac or Δpab	**5 marks**	**Att 2**
$\boldsymbol{|bc|}$	**5 marks**	**Att 2**

5(c)(ii)



$$\cos\angle apc = \frac{20^2 + 20^2 - 38^2}{2(20)(20)} = \frac{-644}{800}$$

$$\therefore \; |\angle apc| = 143.61°$$

or

$$\sin\angle\frac{apc}{2} = \frac{19}{20} \;\Rightarrow\; |\angle apc| = 143.61°.$$

$$\cos\angle apb = \frac{20^2 + 20^2 - 34^2}{2(20)(20)} = \frac{-356}{800}$$

$$\therefore \; |\angle apb| = 116.42°.$$

or

$$\sin\angle\frac{apb}{2} = \frac{17}{20} \;\Rightarrow\; |\angle apb| = 116.42°.$$

$$|\angle cpb| = 360° - (143.61° + 116.42°) = 99.97°.$$

$$|bc|^2 = 20^2 + 20^2 - 2(20)(20)\cos\angle 99.97°$$

$$|bc|^2 = 800 + 138.506 = 938.506$$

$$\therefore \; |bc| = 30.6 \text{ metres.}$$

or

$$\sin\angle\frac{cpb}{2} = \frac{\frac{1}{2}|bc|}{20} \;\Rightarrow\; |bc| = 40\sin\angle\frac{99.97°}{2}$$

$$\therefore \; |bc| = 30.6 \text{ metres.}$$

or

$$\left|\angle pab\right| = \cos^{-1}\frac{17}{20} = 31.788^\circ$$

$$\left|\angle pac\right| = \cos^{-1}\frac{19}{20} = 18.194^\circ$$

Applying cosine rule to triangle *abc*:

$$|bc|^2 = 34^2 + 38^2 - 2(34)(38)\cos 49.982^\circ$$

$$|bc|^2 = 1156 + 1444 - 1661.58 = 938.415$$

$$|bc| = 30.63356$$

$$\Rightarrow |bc| = 30.6 \text{ metres}$$

Blunders (−3)

B1 Error in cosine rule formula (apply once).

B2 Error in substitution into cosine formula.

B3 Incorrect evaluation of angle.

B4 Use of sin but with incorrect ratio of lengths.

Slips (−1)

S1 Arithmetic error.

Attempts (2 marks)

A1 Cosine rule with some substitution.

A2 $\perp$ from p to side and use of sin.

Worthless (0)

W1 Assumes $\angle cpb = 120^0$.

QUESTION 6

Part (a)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**
Part (b)	**30 (5, 5, 10, 5, 5) marks**	**Att (2, 2, 3, 2, 2)**

Part (a)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**
Part (a)(i)	**10 marks**	**Att 3**

6(a)

Nine friends wish to travel in a car. Only two of them, John and Mary, have licences to drive. Only five people can fit in the car (i.e. the driver and four others).

In how many ways can the group of five people be selected if

(i) both John and Mary are included?

Part (a)(i)	**10 marks**	**Att 3**

6(a)(i)

9 people, select group of 5 with John and Mary included.
$\therefore$ Select 3 from 7.
Solution $= {}^7C_3 = 35$.

Blunders (–3)
B1 ${}^7C_5$ or ${}^9C_3$.

Slips (–1)
S1 Arithmetic error.

Attempts (3 marks)
A1 ${}^9C_5$.

Part (a)(ii) **5 marks** **Att 2**

6(a)(ii)	either John or Mary is included, but not both?

Part (a)(ii) **5 marks** **Att 2**

6(a)(ii)

John included, Mary excluded $\Rightarrow {}^7C_4$
Mary included, John excluded $\Rightarrow {}^7C_4$
$\therefore$ Solution $= 2 \times {}^7C_4 = 70$.

Blunders (–3)
B1 $2 \times {}^8C_4$.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 ${}^7C_4$.

Part (a)(iii) **5 marks** **Att 2**

6(a) Later, another one of the nine friends, Anne, gets a driving licence.

(iii) The next time the journey is made, in how many ways can the group of five be chosen, given that at least one licensed driver must be included?

Part (a)(iii) **5 marks** **Att 2**

6(a)(iii)

Total number of selections $= {}^9C_5 = 126$.
Number of selections with no driver $= {}^6C_5 = 6$.
$\therefore$ Number of selections with a driver $= 126 - 6 = 120$.

or

${}^3C_1 \times {}^6C_4 + {}^3C_2 \times {}^6C_3 + {}^3C_3 \times {}^6C_2$
$= 45 + 60 + 15 = 120$.

Blunders (–3)
B1 ${}^9C_5$ and ${}^6C_5$ not subtracted.
B2 One part of 'second' solution omitted, e.g. ${}^3C_1 \times {}^6C_4$.
B3 Error in evaluating ${}^nC_r$.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 ${}^9C_5$ or ${}^6C_5$.
A2 ${}^3C_1 \times {}^6C_4$.

Part (b)	**30 (5, 5, 10, 5, 5) marks**	**Att (2, 2, 3, 2, 2)**

Part (b)(i)	**20 (5, 5, 10) marks**	**Att (2, 2, 3)**

6(b)(i)

Solve the difference equation $6u_{n+2} - 5u_{n+1} + u_n = 0$, where $n \geq 0$, given that $u_0 = 5$ and $u_1 = 2$.

Characteristic equation	**5 marks**	**Att 2**
Characteristic roots	**5 marks**	**Att 2**
Final solution	**10 marks**	**Att 3**

6(b)(i)

$$6u_{n+2} - 5u_{n+1} + u_n = 0, \quad \text{where } n \geq 0$$

$$6x^2 - 5x + 1 = 0$$

$$(2x-1)(3x-1) = 0 \quad \Rightarrow \quad 2x - 1 = 0 \text{ or } 3x - 1 = 0$$

$$\therefore \; x = \frac{1}{2} \quad \text{or} \quad x = \frac{1}{3}.$$

$$u_n = k\left(\frac{1}{2}\right)^n + l\left(\frac{1}{3}\right)^n.$$

$$u_0 = 5 \quad \Rightarrow \quad k + l = 5 \quad \Rightarrow \quad 2k + 2l = 10$$

$$u_1 = 2 \quad \Rightarrow \quad \tfrac{1}{2}k + \tfrac{1}{3}l = 2 \quad \Rightarrow \quad 3k + 2l = 12$$

$$-k = -2$$

$$k = 2 \text{ and } l = 3.$$

$$u_n = 2\left(\frac{1}{2}\right)^n + 3\left(\frac{1}{3}\right)^n.$$

Blunders (–3)
B1 Error in characteristic equation.
B2 Error in factors or in quadratic formula.
B3 Incorrect use of initial conditions.
B4 Roots in decimal form.

Slips (–1)
S1 Arithmetic error.

Attempts (2, 3 marks)
A1 Correct form of u_n and stops
A2 An equation in k and l.

Worthless (0)
W1 No further marks if roots are complex.

Part (b)(ii)	**5 marks**	**Att 2**

6(b)(ii)	Find an expression in n for the sum of the terms $u_0 + u_1 + u_2 + \cdots + u_n$. (Hint: it is the sum of two geometric series.)

Expression for sum of series	**5 marks**	**Att 2**

6(b)(ii)

$u_1 + u_2 + u_3 + u_4 + \;........\; + u_n$

$$= 2\left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \;.......\; + \left(\frac{1}{2}\right)^n\right] + 3\left[\left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \;.......\; + \left(\frac{1}{3}\right)^n\right]$$

$$= 2\left[\frac{1\left(1-\left(\frac{1}{2}\right)^{n+1}\right)}{1-\frac{1}{2}}\right] + 3\left[\frac{1\left(1-\left(\frac{1}{3}\right)^{n+1}\right)}{1-\frac{1}{3}}\right]$$

$$= 4\left[1 - \frac{1}{2^{n+1}}\right] + \frac{9}{2}\left[1 - \frac{1}{3^{n+1}}\right]$$

Blunders (–3)
B1 Error in formula for sum of geometric series.
B2 Incorrect value for n assigned to formula.
B3 Incorrect value for a or r used in formula.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Formula for sum of geometric series with some substitution.
A2 Correct value for a or r.

Part (b)(iii)	**5 marks**	**Att 2**

6(b)(iii)	Evaluate the sum to infinity of this series (that is: $\sum_{n=0}^{\infty} u_n$).

Evaluate sum to infinity	**5 marks**	**Att 2**

6(b)(iii)

$$\sum_{n=0}^{\infty} u_n = 4 + 4\tfrac{1}{2} = 8\tfrac{1}{2}.$$

Note: $\lim_{n\to\infty}\frac{1}{2^{n+1}} = 0$ and $\lim_{n\to\infty}\frac{1}{3^{n+1}} = 0$

Blunders (–3)
B1 Error in formula or limit.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Formula for sum to infinity of geometric series with some substitution.

QUESTION 7

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5)marks**	**Att (2, 2)**
Part (a)(i)	**5 marks**	**Att 2**
Part (a)(ii)	**5 marks**	**Att 2**

7(a) Two unbiased dice, each with faces numbered from 1 to 6, are thrown.

(i) What is the probability of getting a total equal to 8?

(ii) What is the probability of getting a total less than 8?

7 (a) (i)

Number of favourable outcomes = $\#\{(2, 6),(3, 5),(4, 4),(5, 3),(6, 2)\} = 5$.

Number of possible outcomes = $6 \times 6 = 36$.

Probability (of a total equal to 8) $= \frac{5}{36}$.

or count from table.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	**8**
3	4	5	6	7	**8**	
4	5	6	7	**8**		
5	6	7	**8**			
6	7	**8**				

7 (a) (ii)

Number of favourable outcomes (total < 8)

= 21 (count from table or by listing).

Probability (total less than 8) $= \frac{21}{36}$.

or

Number of totals > 8 = 10 $\Rightarrow$ number of totals < 8 is $36 - 10 - 5 = 21$.

Probability (total less than 8) $= \frac{21}{36}$.

Blunders (–3)

B1 Incorrect number of possible outcomes (each time).

Slips (–1)

S1 Arithmetic error.

Attempts (2 marks)

A1 Correct number of possible outcomes.

A2 Correct number of favourable outcomes.

A3 Relevant table or some listing of favourable outcomes.

A4 Incorrect number of favourable outcomes from some relevant work.

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

Part (b)(i) **10 (5, 5) marks** **Att (2, 2)**

7(b) The table below shows the prices of various commodities in the year 2000, as a percentage of their prices in 1999. These are called *price relatives*. (For example, the price relative for *Food, Drink & Other Goods* is 105, indicating that the cost of these items was 5% greater in 2000 than in 1999.)

The table also shows the weight assigned to each commodity. The weight represents the importance of the commodity to the average consumer.

Commodity	Weight	Price in 2000 as % of price in 1999
Housing	8	110
Fuel and Transport	19	108
Tobacco	5	116
Services	16	105
Clothing & Durable Goods	10	97
Food, Drink & Other Goods	42	105

(i) Calculate the weighted mean of the price relatives in the table.

Σxw evaluated **5 marks** **Att 2**
Final solution **5 marks** **Att 2**

7(b)(i)

w	8	19	5	16	10	42
x	110	108	116	105	97	105
$x.w$	880	2052	580	1680	970	4410

$\therefore \sum w = 100$

$\therefore \sum x.w = 10572$

$$\therefore \text{ Weighted mean } = \frac{\sum x.w}{\sum w} = \frac{10572}{100} = 105.72.$$

Blunders (–3)
B1 Σxw incorrect with relevant work (unless due to slip).
B2 Σw incorrect with relevant work (unless due to slip).

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Σxw correct.
A2 Σw correct.

Worthless (0)
W1 takes $\Sigma w = 6$ or uses Σx for Σw.

Part (b)(ii) **10 (5, 5) marks** **Att (2, 2)**

7(b)(ii)

Calculate, correct to two decimal places, the change in the weighted mean if *Tobacco* is removed from consideration.

New weighted mean **5 marks** **Att 2**
Finish **5 marks** **Att 2**

7(b)(ii)

New $\sum w = 100 - 5 = 95$

New $\sum x.w = 10572 - 580 = 9992$

$\therefore$ New weighted mean $= \dfrac{9992}{95} = 105.18$

$105.72 - 105.18 = 0.54$

$\therefore$ The mean decreases by 0.54.

Blunders (–3)
B1 Σxw incorrect with relevant work (unless due to slip).
B2 Σw incorrect with relevant work (unless due to slip).
B3 Change in weighted mean not calculated.

Slips (–1)
S1 Arithmetic error.
S2 Change in weighted mean not correct to two places of decimals.

Attempts (2 marks)
A1 xw correct at least once.
A2 Σw correct.

Worthless (0)
W1 $\Sigma w = 5$ or uses Σx for Σw.

Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (c)(i)	**5 marks**	**Att 2**

7(c)
A palindromic number is one that reads the same backwards as forwards, such as 727 or 38183.

(i) This year, 2002, is a palindromic year. When is the next palindromic year?

Part (c)(i)	**5 marks**	**Att 2**

7(c)(i)
Next palindromic year is 2112.

Blunders (–3)
B1 A palindromic year after 2112 other than 38183.

Attempts (2 marks)
A1 Any correct palindromic year before 2002 other than 727.

Part (c)(ii)	**5 marks**	**Att 2**

7(c) (ii) How many palindromic years are there from 1000 to 9999 inclusive?

From 1000 to 9999 inclusive	**5 marks**	**Att 2**

7(c)(ii)	First digit	Second digit	Third digit	Fourth digit
	Any digit $\neq 0$	any digit	match 2nd. digit	match 1st. digit
	9	10	1	1

$\therefore 9 \times 10 \times 1 \times 1 = 90$

Blunders (–3)
B1 Total = $9\times9\times1\times1$.
B2 Total = $10\times10\times1\times1$.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Incomplete listing with at least one palindromic number identified within range.

Part (c)(iii) **10 (5, 5) marks** **Att (2, 2)**

7(c)(iii)	A whole number, greater than 9 and less than 10 000, is selected at random. What is the probability that the number is palindromic?

Two or three digit palindromic numbers **5 marks** **Att 2**
Final solution **5 marks** **Att 2**

7(c)(iii)

Number of two digit palindromic numbers $= 9 \times 1 = 9$.
Number of three digit palindromic numbers $= 9 \times 10 \times 1 = 90$.
Number of four digit palindromic numbers $= 9 \times 10 \times 1 \times 1 = 90$.
$\therefore$ Total $= 189$.

Number of possible numbers greater than 9 and less than 10 000 = 9990.

$$\therefore \text{ Probability } = \frac{189}{9990} = \frac{7}{370}.$$

Blunders (–3)
B1 Incorrect total from listing.
B2 Two digit palindromic years = 10×1.
B3 Three digit palindromic years = 9×9×1.
B4 Incorrect number of possible outcomes.
B5 Probability not given.

Slips (–1)
S1 Arithmetic error.

Misreadings (–1)
M1 Possible outcomes = 9991.

Attempts (2 marks)
A1 Incomplete listing with at least one palindromic number identified within range.
A2 Correct number of favourable outcomes.
A3 Correct number of possible outcomes.

QUESTION 8

Part (a)	**10 marks**	**Att 3**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 marks**	**Att 3**

8(a) Use integration by parts to find $\int x \ln x dx$.

Integration	**10 marks**	**Att 3**

8(a)

$$\int x \ln x dx = uv - \int v du.$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx \quad ; \quad dv = xdx \Rightarrow v = \int xdx = \frac{1}{2}x^2.$$

$$\therefore \int x \ln x dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 . \frac{1}{x}dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \text{constant.}$$

Blunders (–3)
B1 Incorrect differentiation or integration.
B2 Constant of integration omitted.
B3 Incorrect ‘parts’ formula.

Slips (–1)
S1 Arithmetic error.

Attempts (3 marks)
A1 Correct assigning to parts formula.
A2 Correct differentiation or integration.

Worthless (0)
W1 $u = x$, $dv = \ln x$ with no progress.

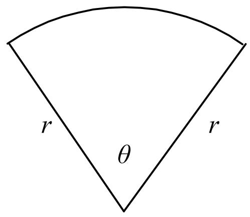
Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

Part (b)(i) **5 marks** **Att 2**

8(b)

The perimeter of a sector of a circle of radius r is 8 metres.

(i) Express θ in terms of r, where θ is the angle of the sector in radians, as shown.



Express θ in terms of r **5 marks** **Att 2**

8(b)(i)

Perimeter = 8 metres

$$2r + r\theta = 8 \quad \Rightarrow \quad r\theta = 8 - 2r$$

$$\theta = \frac{8-2r}{r} = \frac{8}{r} - r.$$

Blunders (–3)
B1 Incorrect formula for length of arc.
B2 θ not expressed in terms of r.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 $2r + r\theta = 8$ and stops.

Part (b)(ii) **5 marks** **Att 2**

8(b)(ii) Hence, show that the area of the sector, in square metres, is $4r - r^2$.

Area of sector **5 marks** **Att 2**

8(b)(ii)

$$\text{Area of sector} = A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{8-2r}{r}\right)$$

$$\therefore A = 4r - r^2.$$

Blunders (–3)
B1 Error in area formula.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Correct substitution into area formula.

Part (b)(iii)	**10 (5, 5) marks**	**Att (2, 2)**

8(b)(iii) Find the maximum possible area of the sector.

Establish $r = 2$	**5 marks**	**Att 2**
Maximum area	**5 marks**	**Att 2**

8(b)(iii)

$$A = 4r - r^2 \;\Rightarrow\; \frac{dA}{dr} = 4 - 2r.$$

$$\frac{dA}{dx} = 0 \;\Rightarrow\; 4 - 2r = 0$$

$$\therefore\; r = 2.$$

For $r = 2\text{m}$, maximum area $= A = 4\text{m}^2$.

$$\left[\text{Note: } \frac{d^2A}{dr^2} = -2 < 0 \;\Rightarrow \text{maximum.}\right]$$

or

$$4r - r^2 = 4 - (r^2 - 4r + 4)$$

$$= 4 - (r-2)^2.$$

Maximum value for $r - 2 = 0$

$\therefore$ Maximum $= 4$.

Blunders (–3)
B1 Error in differentiation.
B2 $4 - (r^2 - 4r + 4)$ and stops.
B3 Maximum area not evaluated.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Correct differentiation.
A2 $\frac{dA}{dr} = 0$ and stops.

Note

No marks awarded if candidate has not shown area is $4r - r^2$ and uses that area now.

Part (c) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

Part (c)(i) **5 marks** **Att 2**

8(c)

The Maclaurin series for $\tan^{-1} x$ is $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

The series is convergent when $|x| < 1$.

(i) Write down the first four terms in the series expansion for $\tan^{-1} \frac{1}{2}$.

First four terms of $\tan^{-1}x$ **5 marks** **Att 2**

8(c)(i)

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\tan^{-1} \frac{1}{2} = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \frac{\left(\frac{1}{2}\right)^7}{7} + \ldots..$$

$$= \frac{1}{2} - \frac{1}{24} + \frac{1}{160} - \frac{1}{896} + \ldots..$$

Blunders (–3)

B1 Incorrect substitution.

Slips (–1)

S1 Arithmetic error.

Attempts (2 marks)

A1 ½ substituted into some terms.

Part (c)(ii) **10 marks (5, 5)** **Att (2, 2)**

8(c)(ii) Use the fact that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$ to derive a series expansion for π, giving the terms up to and including seventh powers.

Expansion of $\tan^{-1}\frac{1}{3}$ **5 marks** **Att 2**

Expansion for π **5 marks** **Att 2**

8(c)(ii)

$$\tan^{-1}\frac{1}{3} = \frac{1}{3} - \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} - \frac{\left(\frac{1}{3}\right)^7}{7} + \ldots..$$

$$= \frac{1}{3} - \frac{1}{81} + \frac{1}{1215} - \frac{1}{15309} + \ldots..$$

$$\pi = 4\left[\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right]$$

$$\pi = 4\left[\frac{1}{2} - \frac{1}{24} + \frac{1}{160} - \frac{1}{896} + \frac{1}{3} - \frac{1}{81} + \frac{1}{1215} - \frac{1}{15309}\right]$$

Blunders (–3)

B1 Incorrect substitution.

B2 Not all terms included.

B3 Not expressed as $\pi = \ldots\ldots$

Slips (–1)

S1 Arithmetic error.

Attempts (2 marks)

A1 $\frac{1}{3}$ substituted into some terms.

Worthless (0)

W1 $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}1$ and substitutes $x = 1$.

Part (c)(iii) **5 marks** **Att 2**

8(c)(iii) Use these terms to find an approximation for π. Give your answer correct to four decimal places.

Approximation of π **5 marks** **Att 2**

8(c)(iii) $\pi = 4[0.787521] = 3.1409.$

Blunders (–3)

B1 Error in evaluation of x^3, x^5, x^7.

Slips (–1)

S1 Arithmetic error.

Attempts (2 marks)

A1 Some evaluation.

QUESTION 9

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (10, 5, 5) marks**	**Att (3, 2, 2)**
Part (c)	**20 (5, 5, 5, 5)marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

9(a) z is a random variable with standard normal distribution.
Find $P(z < -0.46)$.

$P(z < -0.46) = 1 - P(z \leq 0.46)$	**5 marks**	**Att 2**
Final solution	**5 marks**	**Att 2**

9(a)

$$\begin{aligned} &P(z < -0.46) \\ &= 1 - P(z \leq 0.46) \\ &= 1 - 0.6772 \\ &= 0.3228 \end{aligned}$$

Blunders (–3)
B1 Incorrect area defined.
B2 Incorrect reading of tables.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Area required correctly on normal curve.
A2 $P(z \leq 0.46)$.

Part (b) **20 (10, 5, 5) marks** **Att (3, 2, 2)**

Part (b)(i) **10 marks** **Att 3**

9(b) A certain player takes 25 penalty shots during this year's season. Each penalty shot is independent of all others. Experience from previous seasons indicates that on each occasion the probability that this player scores is $\frac{3}{5}$.

(i) Find the probability that she scores exactly 15 of the 25 times.

Find probability **10 marks** **Att 3**

9(b)(i)

$$n = 25, \quad r = 15, \quad p = \frac{3}{5}, \quad q = \frac{2}{5}.$$

$$\text{Probability} = {}^{n}C_{r}\, p^{r} q^{n-r} = {}^{25}C_{15}\left(\frac{3}{5}\right)^{15}\left(\frac{2}{5}\right)^{10} \approx 0.161.$$

Blunders (–3)

B1 Error in binomial.

B2 Incorrect q.

Slips (–1)

S1 Arithmetic error.

Attempts (3 marks)

A1 Use of binomial.

A2 $\left(\frac{3}{5}\right)^{15}$.

Part (b)(ii) **10 (5, 5) marks** **Att (2, 2)**

9(b)(ii) Use the normal approximation to the binomial distribution to estimate the probability that she scores at least 18 times.

Correct $\bar{x}$ and σ	**5 marks**	**Att 2**
Final solution	**5 marks**	**Att 2**

9(b)(ii)

$$\bar{x} = np = 25\left(\frac{3}{5}\right) = 15 \quad ; \quad \sigma = \sqrt{npq} = \sqrt{25\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)} = \sqrt{6}$$

$x \geq 18$ in the binomial corresponds to $x \geq 17.5$ in the normal (continuity correction)

$$z = \frac{x - \bar{x}}{\sigma} = \frac{17.5 - 15}{\sqrt{6}} = 1.02.$$

$$P(x \geq 17.5) = P(z \geq 1.02)$$
$$= 1 - P(z < 1.02)$$
$$= 1 - 0.8461 = 0.1539.$$

Blunders (–3)
B1 Incorrect formula for $\bar{x}$ or σ.
B2 No continuity correction [has $P(x \geq 18)$ when $\bar{x} = 15$].
B3 Error in determining area required.
B4 Incorrect reading from tables.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 $\bar{x} = np$ or $\sigma = \sqrt{npq}$.
A2 $z = \dfrac{x - \bar{x}}{\sigma}$.
A3 A correct step in outlining area required.

Part (c)	**20 (5, 5, 5, 5)marks**	**Att (2, 2, 2, 2)**

Part (c)(i)	**5 marks**	**Att 2**

9(c)(i) $P(E \mid F)$ denotes the conditional probability of "E given F".
Write down an equation to express the relationship between $P(F)$, $P(E \mid F)$ and $P(E \cap F)$.

Part (c)(i)	**5 marks**	**Att 2**

9(c)(i)

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} \quad \text{or} \quad P(E \cap F) = P(F).P(E \mid F).$$

Blunders (–3)

B1 $P(E \mid F) = \dfrac{P(E \cap F)}{P(E)}$.

B2 $P(E \cap F) = P(F).P(F \mid E)$.

Attempts (2 marks)

A1 $P(E).P(E \mid F) = P(E \cup F)$.

Part (c) (ii)	**10 (5, 5) marks**	**Att (2, 2)**

9(c)(ii) E and F are events such that $P(E \mid F) = \frac{1}{2}$, $P(F \mid E) = \frac{1}{3}$, and $P(E \cap F) = \frac{1}{7}$.
Find $P(E \cup F)$.

$P(E)$ or $P(F)$ correct **5 marks** **Att 2**
To finish **5 marks** **Att 2**

9(c)(ii)

$$P(E \mid F) = \frac{1}{2}, \quad P(F \mid E) = \frac{1}{3}, \text{ and } P(E \cap F) = \frac{1}{7}$$

$$P(E \cap F) = P(F).P(E \mid F)$$
$$\frac{1}{7} = P(F).\frac{1}{2} \quad \Rightarrow \quad P(F) = \frac{2}{7}.$$

$$P(E \cap F) = P(E).P(F \mid E)$$
$$\frac{1}{7} = P(E).\frac{1}{3} \quad \Rightarrow \quad P(E) = \frac{3}{7}.$$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$
$$= \frac{3}{7} + \frac{2}{7} - \frac{1}{7}$$
$$\therefore P(E \cup F) = \frac{4}{7}.$$

Blunders (–3)
B1 Theory error in calculating $P(E)$ or $P(F)$.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 $P(E \cap F) = P(E).P(F \mid E)$ or $P(E \cap F) = P(F).P(E \mid F)$.
A2 $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
A3 $P(E \cup F) = P(E) + P(F)$.

Part (c) (iii) **5 marks** **Att 2**

9(c)(iii) Are the events E and F in part **(ii)** independent? Give a reason for your answer.

Are events E and F independent? **5 marks** **Att 2**

9(c)(iii) E and F are not independent events as :
$P(E \mid F) \neq P(E)$ **or** $P(F \mid E) \neq P(F)$ **or** $P(E \cap F) \neq P(E).P(F)$.

Blunders (–3)
B1 $P(E \mid F) \neq P(F)$.

Attempts (2 marks)
A1 Answer NO without reason, given some work of value in part **(c)(ii).**

QUESTION 10

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 10, 5) marks**	**Att (2, 3, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

10(a) The set {0, 2, 4, 6} is a group under addition modulo 8.

Draw up its Cayley table and write down the inverse of each element.

Cayley table	**5 marks**	**Att 2**
Inverses	**5 marks**	**Att 2**

10(a)

+ mod 8	0	2	4	6
0	0	2	4	6
2	2	4	6	0
4	4	6	0	2
6	6	0	2	4

Inverses
$0^{-1} = 0$
$2^{-1} = 6$
$4^{-1} = 4$
$6^{-1} = 2$

Blunders (–3)
B1 One incorrect entry in Cayley table.
B2 All inverses not given.

Attempts (2 marks)
A1 Incomplete Cayley table.
A2 One inverse given.

Part (b) **20 marks (5, 5, 5, 5)** **Att (2, 2, 2, 2)**

Part (b)(i) **5 marks** **Att 2**

10(b)(i)

The incomplete table shown is the Cayley table for the group $\{a, b, c, d\}, *$.

(i) Explain why b must be the identity element.

*	a	b	c	d
a	c			
b				
c			b	
d				c

***b* identity element** **5 marks** **Att 2**

10(b)(i)

$a * a = c \Rightarrow$ a is not the identity element.

$c * c = b \Rightarrow$ c is not the identity element.

$d * d = c \Rightarrow$ d is not the identity element.

$\therefore$ b must be the identity element, as $\{a, b, c, d\}, *$ is a group.

Blunders (–3)

B1 one statement missing, e.g. $d * d = c \Rightarrow d$ not identity.

Attempts (2 marks)

A1 $a * a = c \Rightarrow a$ not identity.

Part (b)(ii) **10 marks (5, 5)** **Att (2, 2)**

10(b)(ii) Copy and complete the table.

***b* row and column correct** **5 marks** **Att 2**

Complete rest of table **5 marks** **Att 2**

10(b)(ii)

*	a	b	c	d
a	c	a	d	b
b	a	b	c	d
c	d	c	b	a
d	b	d	a	c

Blunders (–3)

B1 b row and column not completed correctly.

B2 Rest of table not completed correctly.

Attempts (2 marks)

A1 A correct entry in b row or column.

A2 A correct new entry in other rows or columns.

Part (b) (iii) **5 marks** **Att 2**

10(b)(iii)	List all of the subgroups of $\{a,b,c,d\},*$.

List subgroups **5 marks** **Att 2**

10(b)(iii) Group is of order four.
By Lagrange, the subgroups can only be of order 1, 2 or 4.

Subgroup of order one (improper subgroup) must contain identity i.e. $\{b\}$
Subgroup of order two must contain identity and element of order two i.e. $\{b, c\}$
Subgroup of order four (improper subgroup) is the group itself i.e. $\{a, b, c, d\}$

$\therefore$ subgroups are $\{b\}$, $\{b, c\}$, $\{a, b, c, d\}$.

Blunders (–3)
B1 One incorrect subgroup.

Attempts (2 marks)
A1 One correct subgroup.
A2 Two incorrect subgroups.

Part (c) **20 marks (5, 10, 5)** **Att (2, 3, 2)**

Part (c)(i) **5 marks** **Att 2**

10(c)(i)	$G,*$ is a group and H is a non-empty subset of G. Give a set of conditions that must be verified in order to show that $H,*$ is a subgroup of $G,*$.

Set of conditions **5 marks** **Att 2**

10(c)(i)

For all $a, b \in H$ then $a * b \in H$. $\therefore$ $H,*$ is closed.

For all $a \in H$ then $a^{-1} \in H$. $\therefore$ Inverses exist and hence identity.

or

For all $a,b \in H$ then $a * b^{-1} \in H$. Closure and inverses.

Blunders (–3)
B1 Inverses not established.
B2 Closure not established.

Attempts (2 marks)
A1 Closure only established.
A2 Inverses only established.

Part (c)(ii)	**10 marks**	**Att 3**

10(c)(ii) G is a group and $g \in G$. Prove that the set $H = \{g^n \mid n \in \mathbf{Z}\}$ is a subgroup of G.

Prove the set *H* is a subgroup of *G*	**10 marks**	**Att 3**

10(c)(ii)

Let $g^x, g^y \in H$ for $x, y \in Z$.

$g^x . g^y = g^{x+y}$. But $x + y \in Z$ as Z is closed under addition.

$\therefore \; g^{x+y} \in H \Rightarrow H$ is closed.

$\left(g^x\right)^{-1} = g^{-x}$. But $x \in Z \Rightarrow -x \in Z$.

$\therefore \; g^{-x} \in H \Rightarrow$ each element of H has an inverse.

$\therefore \; H$ is a subgroup of G.

Blunders (–3)
B1 Closure not fully established.
B2 Inverses not fully established.

Attempts (3 marks)
A1 $g^x.g^y$
A2 $(g^x)^{-1}$.

Part (c)(iii)	**5 marks**	**Att 2**

10(c)(iii) C is a cyclic group of order 10 and x is a generator of C.
Describe all the subgroups of C in terms of x.

Describe subgroups in terms of *x*	**5 marks**	**Att 2**

10(c)(iii)

C is of order 10.
By Lagrange, order of subgroups is a factor of 10.
i.e. of order 1, 2, 5, 10.

$\therefore$ Four subgroups generated by x, x^2, x^5, x^{10}.

Blunders (–3)
B1 One subgroup missing.
B2 Incorrect subgroup and three correct subgroups.

Attempts (2 marks)
A1 One correct subgroup given.
A2 Order of a subgroup stated.

QUESTION 11

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**
Part (b)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**
Part (c)	**20 (5, 5, 5, 5) marks**	**Att (2, 2, 2, 2)**

Part (a)	**10 (5, 5) marks**	**Att (2, 2)**

11(a) The equation of an ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$.
Calculate the eccentricity of the ellipse.

$\boldsymbol{b^2 = a^2(1-e^2)}$	**5 marks**	**Att 2**
Correct *e*	**5 marks**	**Att 2**

11(a)

$$a^2 = 25, \quad b^2 = 9.$$

$$b^2 = a^2(1-e^2) \quad \Rightarrow \quad 9 = 25(1-e^2)$$

$$e^2 = 1 - \frac{9}{25} = \frac{16}{25}.$$

$$\therefore e = \frac{4}{5}.$$

Blunders (–3)
B1 Value of a or b incorrect.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 $a^2 = 25$ or $b^2 = 9$.
A2 b incorrect, e.g. $b = a^2(1-e^2)$.
A3 $b^2 = a^2(1+e^2)$.

Part (b) **20 (5, 5, 5, 5) marks** **Att (2, 2, 2, 2)**

Part (b)(i) **5 marks** **Att 2**

11(b)(i)

Let f be the transformation $(x, y) \to (x', y')$, where

$$x' = 3x + 4y + 1$$
$$y' = 4x - 3y + 2.$$

Let $p(x_1, y_1)$ and $q(x_2, y_2)$ be two distinct points.

Find the distance between $f(p)$ and $f(q)$ in terms of x_1, x_2, y_1 and y_2.

$|f(p)\,f(q)|$ **5 marks** **Att 2**

11 (b) (i)

$$p(x_1, y_1) \quad \therefore \quad f(p) = (3x_1 + 4y_1 + 1,\ 4x_1 - 3y_1 + 2)$$
$$q(x_2, y_2) \quad \therefore \quad f(q) = (3x_2 + 4y_2 + 1,\ 4x_2 - 3y_2 + 2)$$
$$|f(p)f(q)| = \sqrt{[3(x_1 - x_2) + 4(y_1 - y_2)]^2 + [4(x_1 - x_2) - 3(y_1 - y_2)]^2}.$$

Blunders (–3)
B1 Error in distance formula.
B2 $f(p)$ or $f(q)$ incorrect.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 $f(p)$ or $f(q)$ correct.
A2 Distance formula with some correct substitution.

Note Accept $|f(p)\,f(q)|$ in non-simplified form.

Part (b)(ii) **15 (5, 5, 5) marks** **Att (2, 2, 2)**

11(b)(ii)	Hence, or otherwise, prove that *f* is a similarity transformation.

$\lvert f(p)\, f(q) \rvert$ simplified	**5 marks**	**Att 2**
Reduced to $\sqrt{k^2 \lvert pq \rvert}$	**5 marks**	**Att 2**
Conclusion	**5 marks**	**Att 2**

11(b)(ii)

$$\lvert f(p) f(q) \rvert = \sqrt{[3(x_1 - x_2) + 4(y_1 - y_2)]^2 + [4(x_1 - x_2) - 3(y_1 - y_2)]^2}$$

$$= \sqrt{25(x_1 - x_2)^2 + 25(y_1 - y_2)^2}$$

$$= 5\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= 5\lvert pq \rvert .$$

$\therefore$ f is a similarity transformation.

Blunders (–3)
B1 Cancellation not made.
B2 Terms not regrouped to $\sqrt{k^2 \lvert pq \rvert}$.
B3 $\lvert pq \rvert$ found but final conclusion not stated.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Some simplification or reduction made.
A2 $\lvert pq \rvert$ found.

Part (c)	20 marks (5, 5, 5, 5)	Att (2, 2, 2, 2)
Part (c)(i)	**5 marks**	**Att 2**

11(c)

$[u'v']$ is a chord of the ellipse $E: \dfrac{x^2}{100} + \dfrac{y^2}{25} = 1$.

The midpoint of $[u'v']$ is $p'(8, 2)$.

(i) Write down a linear transformation *f* that maps the unit circle $S: x^2 + y^2 = 1$ onto *E*.

Linear transformation	5 marks	Att 2

11(c)(i)

$$f: (x, y) \rightarrow (10x, 5y) \text{ or matrix of transformation } f = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}.$$

Blunders (–3)

B1 Matrix of *f* with main diagonal correct but minor diagonal incorrect.

Attempts (2 marks)

A1 Incorrect transformation but with $10x$ or $5y$.

Part (c(ii)	5 marks	Att 2

11(c)(ii) Write down the co-ordinates of *p*, where $f(p) = p'$.

Co-ordinates of *p*	5 marks	Att 2

11(c)(ii)

$$10x = 8 \Rightarrow x = \frac{4}{5}, \quad 5y = 2 \Rightarrow y = \frac{2}{5}. \qquad \therefore p\left(\frac{4}{5}, \frac{2}{5}\right).$$

or

$$p = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} 8 \\ 2 \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 40 \\ 20 \end{pmatrix} \qquad \therefore p\left(\frac{4}{5}, \frac{2}{5}\right).$$

Blunders (–3)

B1 Error in inverse matrix.

B2 Error in matrix multiplication.

Slips (–1)

S1 Arithmetic error.

Attempts (2 marks)

A1 One component of point *p* found.

A2 $10x = 8$ or $5y = 2$.

Part (c) (iii)	**5 marks**	**Att 2**

11(c)(iii)	Noting that, in a circle, the line joining the centre to the midpoint of a chord is perpendicular to the chord, find the equation of *uv*, where $f(u) = u'$ and $f(v) = v'$.

Equation of *uv*	**5 marks**	**Att 2**

11(c)(iii)

Centre of circle S is $(0,\ 0)$. $p\left(\frac{4}{5},\ \frac{2}{5}\right)$ is midpoint of chord $[uv]$.

Slope $op = \frac{\frac{2}{5}}{\frac{4}{5}} = \frac{1}{2} \Rightarrow$ slope $uv = -2$.

equation of uv : $y - \frac{2}{5} = -2\left(x - \frac{4}{5}\right)$

uv : $10x + 5y = 10 \Rightarrow 2x + y = 2$.

Blunders (–3)
B1 Error in slope formula.
B2 Error in perpendicular slope.
B3 Error in equation of line.
B4 Incorrect point substituted into line equation.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Slope *op*.

Part (c)(iv) **5 marks** **Att 2**

11(c)(iv) Find the co-ordinates of u and v, and hence the co-ordinates of u' and v'.

Find the co-ordinates of *u* and *v* **5 marks** **Att 2**

11(c)(iv)

$$2x + y = 2 \cap x^2 + y^2 = 1.$$

$$y = -2x + 2 \quad \Rightarrow \quad x^2 + (-2x+2)^2 = 1$$

$$5x^2 - 8x + 3 = 0 \quad \Rightarrow \quad (x-1)(5x-3) = 0$$

$$\therefore \ x = 1 \ \text{ or } \ x = \frac{5}{3}$$

$$\therefore \ u\,(1,\ 0) \ \text{ and } \ v\left(\frac{3}{5},\ \frac{4}{5}\right).$$

$$u' = f(1,\ 0) \quad \Rightarrow \quad u' = (10,\ 0)$$

$$v' = f\left(\frac{3}{5},\ \frac{4}{5}\right) \quad \Rightarrow \quad v' = (6,\ 4).$$

Blunders (–3)
B1 Error in squaring.
B2 Error in factors.

Slips (–1)
S1 Arithmetic error.

Attempts (2 marks)
A1 Solving between line and circle.
A2 Correct quadratic.
A3 Finds *u* and *v* and stops.