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LEAVING CERTIFICATE EXAMINATION, 2001

MATHEMATICS — HIGHER LEVEL

PAPER 1 (300 marks)

THURSDAY, 7 JUNE — MORNING, 9.30 to 12.00

Attempt **SIX QUESTIONS** (50 marks each).

WARNING: Marks may be lost if all necessary work is not clearly shown.

1. **(a)** Find the real numbers a and b such that

$$x^2 + 4x - 6 = (x+a)^2 + b \quad \text{for all } x \in \mathbf{R}.$$

(b) Let $f(x) = 2x^3 + mx^2 + nx + 2$ where m and n are constants.

Given that $x-1$ and $x+2$ are factors of $f(x)$, find the value of m and the value of n.

(c) $x^2 - px + q$ is a factor of $x^3 + 3px^2 + 3qx + r$.

(i) Show that $q = -2p^2$.

(ii) Show that $r = -8p^3$.

(iii) Find the three roots of $x^3 + 3px^2 + 3qx + r = 0$ in terms of p.

2. **(a)** Solve the simultaneous equations

$$x - y = 0$$
$$(x+2)^2 + y^2 = 10.$$

(b) **(i)** Solve for x

$$|3x+5| < 4.$$

(ii) Simplify $\left(x^2 + \sqrt{2} + \frac{1}{x^2}\right)\left(x^2 - \sqrt{2} + \frac{1}{x^2}\right)$ and express your answer in the form $x^n + \frac{1}{x^n}$ where n is a whole number.

(c) α and β are real numbers such that $\alpha + \beta = -7$ and $\alpha\beta = 11$.

(i) Show that $\alpha^2 + \beta^2 = 27$.

(ii) Find a quadratic equation with roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ and write your answer in the form $px^2 + qx + r = 0$ where $p, q, r \in \mathbf{Z}$.

3. **(a)** Let $u = \dfrac{1+3i}{3+i}$ where $i^2 = -1$.

(i) Express u in the form $a+ib$ where $a,b \in \mathbf{R}$.

(ii) Evaluate $|u|$.

(b) **(i)** Write the simultaneous equations

$$x - \sqrt{3}y = -2$$
$$\sqrt{3}x + y = 2\sqrt{3}$$

in the form $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix}$ where A is a 2×2 matrix.

(ii) Then, find A^{-1} and use it to solve the equations for x and y.

(c) **(i)** Write $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -2 & 3 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ in the form $ax^2 + bxy + cy^2$ where $a,b,c \in \mathbf{Z}$.

(ii) Show that $z^2 - 16$ is a factor of $z^3 + (1+i)z^2 - 16z - 16(1+i)$ and hence, find the three roots of $z^3 + (1+i)z^2 - 16z - 16(1+i) = 0$.

4. **(a)** The sum of the first n terms of an arithmetic series is given by $S_n = 3n^2 - 4n$.

Use S_n to find **(i)** the first term, T_1

(ii) the sum of the second term and the third term, $T_2 + T_3$.

(b) **(i)** Show that $\dfrac{1}{(n+2)(n+3)} = \dfrac{1}{n+2} - \dfrac{1}{n+3}$ for $n \in \mathbf{N}$.

(ii) Hence, find $\displaystyle\sum_{n=1}^{k} \frac{1}{(n+2)(n+3)}$ and evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$.

(c) **(i)** Write $\dfrac{n^3+8}{n+2}$ in the form $an^2 + bn + c$ where $a,b,c \in \mathbf{R}$.

(ii) Hence, evaluate $\displaystyle\sum_{n=1}^{30} \frac{n^3+8}{n+2}$.

[Note: $\displaystyle\sum_{n=1}^{k} n = \frac{k}{2}(k+1)$; $\displaystyle\sum_{n=1}^{k} n^2 = \frac{k}{6}(k+1)(2k+1)$.]

5. **(a)** The second term, T_2, of a geometric sequence is 21.
The third term, T_3, is -63.

Find **(i)** the common ratio

(ii) the first term.

(b) **(i)** Solve $\log_6(x+5) = 2 - \log_6 x$ for $x > 0$.

(ii) In the binomial expansion of $(1+kx)^6$, the coefficient of x^4 is 240.

Find the two possible real values of k.

(c) Use induction to prove that, for n a positive integer,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all $\theta \in \mathbf{R}$ and $i^2 = -1$.

6. **(a)** Differentiate $\dfrac{x}{1+x^2}$ with respect to x.

(b) **(i)** Given that $y = \sqrt{x}$, what is $\dfrac{dy}{dx}$?

(ii) Now, find from first principles the derivative of $\sqrt{x}$ with respect to x.

(c) Let $x = t^2e^t$ and $y = t + 2\ln t$ for $t > 0$.

(i) Find $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ in terms of t.

(ii) Hence, show that $\dfrac{dy}{dx} = \dfrac{1}{x}$.

7. **(a)** Taking $x_1 = 1$ as the first approximation to the real root of the equation

$$x^3 + x^2 - 1 = 0,$$

use the Newton-Raphson method to find x_2, the second approximation.

(b) **(i)** Differentiate $\tan^{-1} 7x$ with respect to x.

(ii) Given that $y = \sin x \cos x$, find $\frac{dy}{dx}$ and express it in the form $\cos nx$ where $n \in \mathbf{Z}$.

(c) Let $g(x) = x^2 + \frac{a}{x^2}$ where a is a real number and $x \in \mathbf{R}$, $x \neq 0$.

Given that $g(x)$ has a turning point at $x = 2$,

(i) find the value of a

(ii) prove that $g(x)$ has no local maximum points.

8. **(a)** Find **(i)** $\int \frac{1}{x^3} dx$ **(ii)** $\int \sin 5x dx$.

(b) Evaluate **(i)** $\int_0^3 \frac{12}{x^2+9} dx$ **(ii)** $\int_0^4 \frac{(x+4)}{\sqrt{x^2+8x+1}} dx$.

(c) a is a real number such that $0 < a < 8$.

The line $y = ax$ intersects the curve $y = x(8-x)$ at $x = 0$ and at $x = p$.

(i) Show that $p = 8 - a$.

(ii) Show that the area between the curve and the line is $\frac{p^3}{6}$ square units.