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LEAVING CERTIFICATE EXAMINATION, 2002

MATHEMATICS — HIGHER LEVEL

PAPER 1 (300 marks)

THURSDAY, 6 JUNE — MORNING, 9.30 TO 12.00

Attempt **SIX QUESTIONS** (50 marks each).

WARNING: Marks will be lost if all necessary work is not clearly shown.

1. **(a)** Solve the equation

$$x = \sqrt{x+2}.$$

(b) The cubic equation $x^3 - 4x^2 + 9x - 10 = 0$ has one integer root and two complex roots. Find the three roots.

(c) $(p+r-t)x^2 + 2rx + (t+r-p) = 0$ is a quadratic equation, where p, r, and t are integers.

Show that

(i) the roots are rational

(ii) one of the roots is an integer.

2. **(a)** Solve, without using a calculator, the following simultaneous equations:

$$\begin{aligned} x + 2y + 4z &= 7 \\ x + 3y + 2z &= 1 \\ -y + 3z &= 8. \end{aligned}$$

(b) **(i)** Find the range of values of $x \in \mathbf{R}$ for which

$$x^2 + x - 20 \leq 0.$$

(ii) Let $g(x) = x^n + 3$, for all $x \in \mathbf{R}$, where $n \in \mathbf{N}$.

Show that if n is odd then $g(x) + g(-x)$ is constant.

(c) **(i)** Show that if the roots of $x^2 + bx + c = 0$ differ by 1, then $b^2 - 4c = 1$.

(ii) The roots of the equation

$$x^2 + (4k-5)x + k = 0$$

are consecutive integers.

Using the result from part **(i)**, or otherwise, find the value of k and the roots of the equation.

3. **(a)** Express $-1+\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$.

(b) **(i)** Given that $z = 2 - i\sqrt{3}$, find the real number t such that $z^2 + tz$ is real.

(ii) w is a complex number such that

$$w\overline{w} - 2iw = 7 - 4i,$$

where $\overline{w}$ is the complex conjugate of w.

Find the two possible values of w.
Express each in the form $p + qi$, where $p, q \in \mathbf{R}$.

(c) The following three statements are true whenever x and y are real numbers:

- $x + y = y + x$
- $xy = yx$
- If $xy = 0$ then either $x = 0$ or $y = 0$.

Investigate whether the statements are also true when x is the matrix $\begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}$ and y is the matrix $\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$.

4. **(a)** Find, in terms of n, the sum of the first n terms of the geometric series

$$3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots$$

(b) **(i)** Show that $\dfrac{2}{k(k+2)} = \dfrac{1}{k} - \dfrac{1}{k+2}$, for all $k \in \mathbf{R}$, $k \neq 0, -2$.

(ii) Evaluate, in terms of n, $\displaystyle\sum_{k=1}^{n} \frac{2}{k(k+2)}$.

(iii) Evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+2)}$.

(c) Three numbers are in arithmetic sequence.
Their sum is 27 and their product is 704.

Find the three numbers.

5. **(a)** Find the value of x in each case:

(i) $\frac{8}{2^x} = 32$

(ii) $\log_9 x = \frac{3}{2}$.

(b) The first three terms in the binomial expansion of $(1+ax)^n$ are $1 + 2x + \frac{7}{4}x^2$.

(i) Find the value of a and the value of n.

(ii) Hence, find the middle term in the expansion.

(c) Prove by induction that, for any positive integer n,

$$x + x^2 + x^3 + \ldots + x^n = \frac{x(x^n - 1)}{x-1}, \quad \text{where } x \neq 1.$$

6. **(a)** Differentiate $(x^4+1)^5$ with respect to x.

(b) **(i)** Prove, from first principles, the addition rule:

if $f(x) = u(x) + v(x)$ then $\frac{df}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

(ii) Given $y = 2x - \sin 2x$, find $\frac{dy}{dx}$.

Give your answer in the form $k\sin^2 x$, where $k \in \mathbf{Z}$.

(c) The function $f(x) = ax^3 + bx^2 + cx + d$ has a maximum point at (0, 4) and a point of inflection at (1, 0).

Find the values of a, b, c and d.

7. **(a)** Find the slope of the tangent to the curve

$$9x^2 + 4y^2 = 40 \text{ at the point } (2, 1).$$

(b) **(i)** Given that $y = \sin^{-1} 10x$, evaluate $\frac{dy}{dx}$ when $x = \frac{1}{20}$.

(ii) The parametric equations of a curve are

$$x = \ln(1+t^2) \text{ and } y = \ln 2t, \quad \text{where } t \in \mathbf{R}, t > 0.$$

Find the value of $\frac{dy}{dx}$ when $t = \sqrt{5}$.

(c) Let $f(x) = \frac{e^x + e^{-x}}{2}$.

(i) Show that $f''(x) = f(x)$, where $f''(x)$ is the second derivative of $f(x)$.

(ii) Show that $\frac{f'(2x)}{f'(x)} = 2f(x)$ when $x \neq 0$ and where $f'(x)$ is the first derivative of $f(x)$.

8. **(a)** Find $\int (x^3 + \sqrt{x} + 2)dx$.

(b) Evaluate **(i)** $\int_2^7 \frac{2x-4}{x^2-4x+29}dx$ **(ii)** $\int_2^7 \frac{1}{x^2-4x+29}dx$.

(c) Let $f(x) = x^3 - 3x^2 + 5$.
L is the tangent to the curve $y = f(x)$ at its local maximum point.

Find the area enclosed between L and the curve.