Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE EXAMINATION, 2003

MATHEMATICS — HIGHER LEVEL

PAPER 1 (300 marks)

THURSDAY, 5 JUNE — MORNING, 9:30 to 12:00

Attempt **SIX QUESTIONS** (50 marks each).

WARNING: Marks will be lost if all necessary work is not clearly shown.

1. **(a)** Express the following as a single fraction in its simplest form:

$$\frac{6y}{x(x+4y)} - \frac{3}{2x}.$$

(b) **(i)** $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbf{R}$.

Given that k is a real number such that $f(k) = 0$, prove that $x - k$ is a factor of $f(x)$.

(ii) Show that $2x - \sqrt{3}$ is a factor of $4x^2 - 2(1+\sqrt{3})x + \sqrt{3}$ and find the other factor.

(c) The real roots of $x^2 + 10x + c = 0$ differ by $2p$ where $c, p \in \mathbf{R}$ and $p > 0$.

(i) Show that $p^2 = 25 - c$.

(ii) Given that one root is greater than zero and the other root is less than zero, find the range of possible values of p.

2. **(a)** Solve the simultaneous equations:

$$3x - y = 8$$
$$x^2 + y^2 = 10.$$

(b) **(i)** Solve for x:

$$|4x + 7| < 1.$$

(ii) Given that $x^2 - ax - 3$ is a factor of $x^3 - 5x^2 + bx + 9$ where $a, b \in \mathbf{R}$, find the value of a and the value of b.

(c) **(i)** Solve for y:

$$2^{2y+1} - 5(2^y) + 2 = 0.$$

(ii) Given that $x = \alpha$ and $x = \beta$ are the solutions of the quadratic equation

$2k^2x^2 + 2ktx + t^2 - 3k^2 = 0$ where $k, t \in \mathbf{R}$ and $k \neq 0$,

show that $\alpha^2 + \beta^2$ is independent of k and t.

3. **(a)** Evaluate $\begin{pmatrix} 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(b) **(i)** Given that $z = 2 - i$, calculate $\left| z^2 - z + 3 \right|$ where $i^2 = -1$.

(ii) k is a real number such that $\dfrac{-1 + i\sqrt{3}}{-4\sqrt{3} - 4i} = ki$.

Find k.

(c) $1, \omega, \omega^2$ are the three roots of the equation $z^3 - 1 = 0$.

(i) Prove that $1 + \omega + \omega^2 = 0$.

(ii) Hence, find the value of $(1 - \omega - \omega^2)^5$.

4. **(a)** Express the recurring decimal 0.252525 … in the form $\dfrac{p}{q}$ where $p, q \in \mathbf{N}$ and $q \neq 0$.

(b) In an arithmetic series, the sum of the second term and the fifth term is 18. The sixth term is greater than the third term by 9.

(i) Find the first term and the common difference.

(ii) What is the smallest value of n such that $S_n > 600$, where S_n is the sum of the first n terms of the series?

(c) **(i)** $u_1, u_2, u_3, u_4, u_5, \ldots\ldots$ is a sequence where $u_1 = 2$ and $u_{n+1} = (-1)^n u_n + 3$.

Evaluate u_2, u_3, u_4, u_5 and u_{10}.

(ii) a, b, c, d are the first, second, third and fourth terms of a geometric sequence, respectively.

Prove that $a^2 - b^2 - c^2 + d^2 \geq 0$.

5. **(a)** Solve for x:

$$x = \sqrt{7x-6} + 2.$$

(b) Use induction to prove that 8 is a factor of $7^{2n+1} + 1$ for any positive integer n.

(c) Consider the binomial expansion of $\left(ax + \dfrac{1}{bx}\right)^8$, where a and b are non-zero real numbers.

(i) Write down the general term.

(ii) Given that the coefficient of x^2 is the equal to the coefficient of x^4, show that $ab = 2$.

6. **(a)** Differentiate $\sqrt{1+4x}$ with respect to x.

(b) Show that the equation $x^3 - 4x - 2 = 0$ has a root between 2 and 3.

Taking $x_1 = 2$ as the first approximation to this root, use the Newton-Raphson method to find x_3, the third approximation. Give your answer correct to two decimal places.

(c) The function $f(x) = \dfrac{1}{1-x}$ is defined for $x \in \mathbf{R} \setminus \{1\}$.

(i) Prove that the graph of f has no turning points and no points of inflection.

(ii) Write down a reason that justifies the statement: "f is increasing at every value of $x \in \mathbf{R}\setminus\{1\}$".

(iii) Given that $y = x + k$ is a tangent to the graph of f where k is a real number, find the two possible values of k.

7. **(a)** Differentiate each of the following with respect to x:

(i) $\cos^4 x$ **(ii)** $\sin^{-1}\dfrac{x}{5}$.

(b) **(i)** The parametric equations of a curve are:

$$x = \cos t + t\sin t$$
$$y = \sin t - t\cos t \quad \text{where } 0 < t < \frac{\pi}{2}.$$

Find $\dfrac{dy}{dx}$ and write your answer in its simplest form.

(ii) Given that $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{6}$, find the value of $\dfrac{dy}{dx}$ at the point $(2,-3)$.

(c) **(i)** Given that $y = \ln\dfrac{1+x^2}{1-x^2}$ for $0 < x < 1$,

find $\dfrac{dy}{dx}$ and write your answer in the form $\dfrac{kx}{1-x^k}$ where $k \in \mathbf{N}$.

(ii) Given that $f(\theta) = \sin(\theta + \pi)\cos(\theta - \pi)$, find the derivative of $f(\theta)$ and express it in the form $\cos n\theta$ where $n \in \mathbf{Z}$.

8. **(a)** Find **(i)** $\displaystyle\int (x^3 + 2)\,dx$ **(ii)** $\displaystyle\int e^{7x}\,dx$.

(b) **(i)** Evaluate $\displaystyle\int_0^1 \frac{2x}{\sqrt{1+x^2}}\,dx$.

(ii) By letting $u = \sin x$, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos x \sin^6 x\,dx$.

(c) **(i)** Show that $\displaystyle\int_a^{2a} \sin 2x\,dx = \sin 3a \sin a$.

(ii) Use integration methods to show that the volume of a sphere with radius r is $\dfrac{4}{3}\pi r^3$.

Blank Page