2011. M 29

Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE EXAMINATION, 2011

MATHEMATICS – HIGHER LEVEL

PAPER 1 (300 marks)

FRIDAY, 10 JUNE – AFTERNOON, 2:00 to 4:30

Attempt **SIX QUESTIONS** (50 marks each).

WARNING: **Marks will be lost if all necessary work is not clearly shown.**

Answers should include the appropriate units of measurement, where relevant.

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1. **(a)** Simplify fully $\frac{x+1}{x-1}-\frac{x-1}{x+1}-\frac{4}{x^2-1}$.

(b) **(i)** Prove the factor theorem for polynomials of degree 2.
That is, given that $f(x)=ax^2+bx+c$ and k is a number such that $f(k)=0$,
prove that $(x-k)$ is a factor of $f(x)$.

(ii) The factor theorem also holds for polynomials of higher degree.
Find the values of n for which $(x+k)$ is a factor of the polynomial
$g(x)=x^n+k^n$, where $k \neq 0$.

(c) $(x-a)^2$ is a factor of $2x^3-5ax^2+8abx-36a$, where $a \neq 0$.
Find the possible values of a and b.

2. **(a)** Solve for x: $|2x-1| \leq 3$, where $x \in \mathbb{R}$.

(b) α and $\frac{1}{\alpha}$ are the roots of the quadratic equation $3kx^2-18tx+(2k+3)=0$,
where t and k are constants.

(i) Find the value of k.

(ii) If one of the roots is four times the other, find the possible values of t.

(c) Let $f(x)=\frac{1}{x^2-6x+a}$, where a is a constant.

(i) Prove that if $a=13$, then $f(x)>0$ for all $x \in \mathbb{R}$.

(ii) Prove that if $a=13$, then $f(x)<1$ for all $x \in \mathbb{R}$.

(iii) Find the range of values of a such that $0<f(x)<1$, for all $x \in \mathbb{R}$.

3. **(a)** Express $\frac{1+2i}{2-i}$ in the form of $a+bi$, where $i^2=-1$.

(b) **(i)** Find the two complex numbers $a+bi$ such that

$$(a+bi)^2=-3+4i.$$

(ii) Hence solve the equation

$$x^2+x+1-i=0.$$

(c) **(i)** Let A and B be 2×2 matrices, where A has an inverse.

Show that $\left(A^{-1}BA\right)^n = A^{-1}B^nA$ for all $n\in\mathbb{N}$.

Let $P=\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}$ and $M=\begin{pmatrix}-5 & 3\\ -10 & 6\end{pmatrix}$.

(ii) Evaluate $P^{-1}MP$ and hence $\left(P^{-1}MP\right)^n$.

(iii) Hence, or otherwise, show that $M^n=M$, for all $n\in\mathbb{N}$.

4. **(a)** In an arithmetic sequence, the third term is -3 and the sixth term is -15. Find the first term and the common difference.

(b) Let $u_n=l\left(\frac{1}{2}\right)^n+m(-1)^n$ for all $n\in\mathbb{N}$.

(i) Verify that u_n satisfies the equation $2u_{n+2}+u_{n+1}-u_n=0$.

(ii) If $a_k=u_k+u_{k+1}$, express a_k in terms of k and l.

(iii) Find $\sum_{k=1}^{\infty}a_k$, in terms of l.

(iv) For $l>0$, find the least positive integer n for which

$$\sum_{k=1}^{n}a_k>(0{\cdot}99)\sum_{k=1}^{\infty}a_k.$$

5. **(a)** Find the coefficient of x^8 in the expansion of $(x^2-1)^{10}$.

(b) **(i)** Solve the equation:

$$\log_2 x - \log_2 (x-1) = 4\log_4 2.$$

(ii) Solve the equation:

$$3^{2x+1} - 17\left(3^x\right) - 6 = 0.$$

Give your answer correct to two decimal places.

(c) Prove by induction that 9 is a factor of $5^{2n+1} + 2^{4n+2}$, for all $n \in \mathbb{N}$.

6. **(a)** Differentiate $\cos^2 x$ with respect to x.

(b) The equation of a curve is $y = e^{-x^2}$.

(i) Find $\frac{dy}{dx}$.

(ii) Find the co-ordinates of the turning point of the curve.

(iii) Determine whether this turning point is a local maximum or a local minimum.

(c) The function f is defined as $x \to \frac{2x}{x+1}$, where $x \in \mathbb{R} \setminus \{-1\}$.

(i) Find the equations of the asymptotes of the curve $y = f(x)$.

(ii) P and Q are distinct points on the curve $y = f(x)$.
The tangent at Q is parallel to the tangent at P.
The co-ordinates of P are (1, 1).
Find the co-ordinates of Q.

(iii) Verify that the point of intersection of the asymptotes is the midpoint of $[PQ]$.

7. **(a)** Find the slope of the tangent to the curve $x^2 + y^3 = x - 2$ at the point $(3, -2)$.

(b) A curve is defined by the parametric equations

$$x = \frac{t-1}{t+1} \quad \text{and} \quad y = \frac{-4t}{(t+1)^2}, \quad \text{where } t \neq -1.$$

(i) Find $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

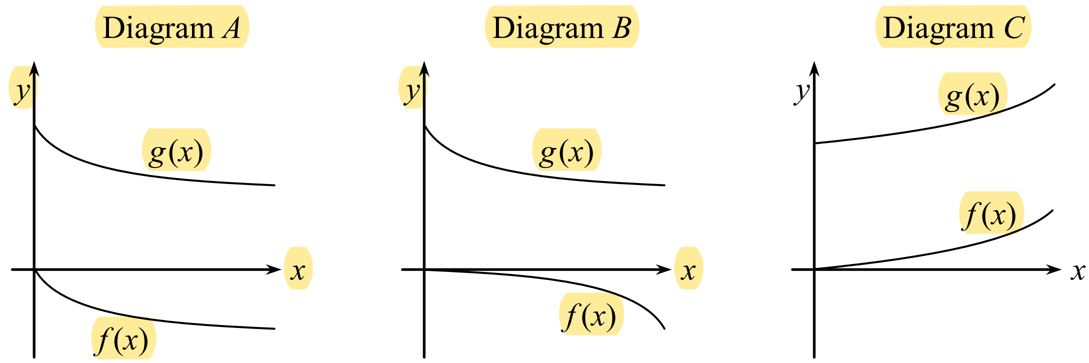
(ii) Hence find $\dfrac{dy}{dx}$, and express your answer in terms of x.

(c) The functions f and g are defined on the domain $x \in \mathbb{R} \setminus \{-1, 0\}$ as follows:

$$f : x \to \tan^{-1}\left(\frac{-x}{x+1}\right) \quad \text{and} \quad g : x \to \tan^{-1}\left(\frac{x+1}{x}\right).$$

(i) Show that $f'(x) = \dfrac{-1}{2x^2 + 2x + 1}$.

(ii) It can be shown that $f'(x) = g'(x)$.
One of the three diagrams A, B, or C below represents parts of the graphs of f and g. Based only on the derivatives, state which diagram is the correct one, and state also why each of the other two diagrams is incorrect.



8. **(a)** Find $\int \left(x^3 + \sqrt{x}\right) dx$.

(b) **(i)** Evaluate $\int_0^2 \frac{x+1}{x^2+2x+2}\, dx$.

(ii) Evaluate $\int_0^2 \frac{x^2+2x+2}{x+1}\, dx$.

(c) Use integration methods to establish the formula $A = \pi r^2$ for the area of a disc of radius r.

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