2011. M30

Coimisiún na Scrúduithe Stáit
State Examinations Commission

LEAVING CERTIFICATE EXAMINATION, 2011

MATHEMATICS – HIGHER LEVEL

PAPER 2 (300 marks)

MONDAY, 13 JUNE – MORNING, 9:30 to 12:00

Attempt **FIVE** questions from **Section A** and **ONE** question from **Section B**.
Each question carries 50 marks.

WARNING: **Marks will be lost if all necessary work is not clearly shown.**

Answers should include the appropriate units of measurement, where relevant.

SECTION A

Answer FIVE questions from this section.

1. **(a)** The following parametric equations define a circle:

$$x = 2 + 3\sin\theta, \quad y = 3\cos\theta, \text{ where } \theta \in \mathbb{R}.$$

What is the Cartesian equation of the circle?

(b) Find the equation of the circle that passes through the points (0, 3), (2, 1) and (6, 5).

(c) The circle $c_1 : x^2 + y^2 - 8x + 2y - 23 = 0$ has centre A and radius r_1.
The circle $c_2 : x^2 + y^2 + 6x + 4y + 3 = 0$ has centre B and radius r_2.

(i) Show that c_1 and c_2 intersect at two points.

(ii) Show that the tangents to c_1 at these points of intersection pass through the centre of c_2.

2. **(a)** Find the value of s and the value of t that satisfy the equation

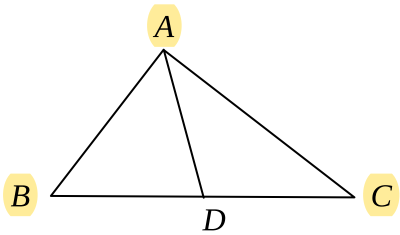
$$s\left(\vec{i} - 4\vec{j}\right) + t\left(2\vec{i} + 3\vec{j}\right) = 4\vec{i} - 27\vec{j}.$$

(b) $\overrightarrow{OP} = 3\vec{i} - 4\vec{j}$ and $\overrightarrow{OQ} = 5\left(\overrightarrow{OP}^{\perp}\right)$, where O is the origin.

(i) Find $\overrightarrow{OQ}$ in terms of $\vec{i}$ and $\vec{j}$.

(ii) Find $\cos\left|\angle OQP\right|$, in surd form.

(c) ABC is a triangle and D is the mid-point of $[BC]$.



(i) Express $\overrightarrow{AB}$ in terms of $\overrightarrow{AD}$ and $\overrightarrow{BC}$ and express $\overrightarrow{AC}$ in terms of $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

(ii) Hence, prove that $\left|AB\right|^2 + \left|AC\right|^2 = 2\left|AD\right|^2 + \frac{1}{2}\left|BC\right|^2$.

3. **(a)** P and Q are the points (–1, 4) and (3, 7) respectively.
Find the co-ordinates of the point that divides $[PQ]$ internally in the ratio 3 : 1.

(b) f is the transformation $(x, y) \to (x', y')$, where $x' = x - y$ and $y' = 2x + 3y$.
l_1 is the line $2x - y - 1 = 0$.

(i) Find the equation of $f(l_1)$, the image of l_1 under f.

(ii) Prove that f maps every pair of parallel lines to a pair of parallel lines.
You may assume that f maps every line to a line.

(iii) The line l_2 is parallel to the line l_1.
$f(l_2)$ intersects the x-axis at A' and the y-axis at B'.
The area of the triangle $A'OB'$ is 9 square units, where O is the origin.
Find the two possible equations of l_2.

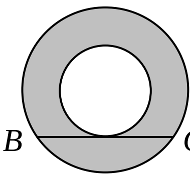
(iv) Given that $A' = f(A)$ and $B' = f(B)$, show that $|\angle AOB| \neq |\angle A'OB'|$.

4. **(a)** Evaluate $\lim\limits_{x \to 0} \left(\dfrac{\sin 2x + \sin x}{3x} \right)$.

(b) Find all the solutions of the equation

$$\sin 2x + \cos x = 0, \text{ where } 0^\circ \leq x \leq 360^\circ.$$

(c) The diagram shows two concentric circles.
A tangent to the inner circle cuts the outer circle at B and C, where $|BC| = 2x$.


B C

(i) Express the area of the shaded region in terms of x.

(ii) In the case where the radius of the outer circle is $2x$, show that the portion of the shaded region that lies below BC has area $\left(\dfrac{2\pi}{3} - \sqrt{3} \right) x^2$.

5. **(a)** Find the values of x for which $3\tan x = \sqrt{3}$, where $0^\circ \le x \le 360^\circ$.

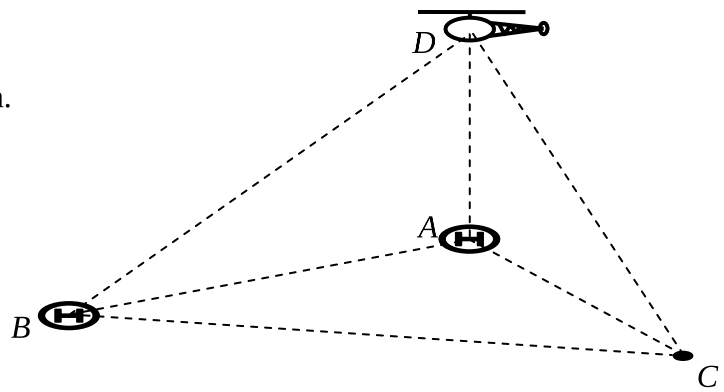
(b) **(i)** Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

(ii) Show that if $\alpha + \beta = 90^\circ$, then $\dfrac{\tan 2\alpha}{\tan 2\beta} = -1$.

(c) A and B are two helicopter landing pads on level ground. C is another point on the same level ground. $|BC| = 800$ metres, $|AC| = 900$ metres, and $|\angle BCA| = 60^\circ$.
A helicopter at point D is hovering vertically above A.
A person at C observes the helicopter to have an angle of elevation of 30°.

(i) Find $|AD|$, in surd form.

(ii) Find $|BD|$.



6. **(a)** Two adults and four children stand in a row for a photograph. How many different arrangements are possible if the four children are between the two adults?

(b) **(i)** Solve the difference equation $u_{n+2} - 6u_{n+1} + 8u_n = 0$, where $n \ge 0$, given that $u_0 = 0$ and $u_1 = 4$.

(ii) For what value of n is $u_n = 30(2^n)$?

(c) Five cards are drawn together at random from a standard pack of 52 playing cards. Find, in decimal form, correct to two significant figures, the probability that:

(i) all five cards are diamonds

(ii) all five cards are of the same suit

(iii) the five cards are the ace, two, three, four and five of diamonds

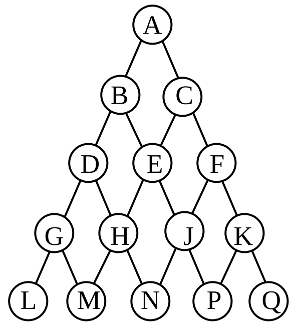
(iv) the five cards include the four aces.

7. **(a)** A team of four is selected from a group of seven girls and five boys.

(i) How many different selections are possible?

(ii) How many of these selections include at least one girl?

(b) A marble falls down from A and must follow one of the paths indicated on the diagram. All paths from A to the bottom row are equally likely to be followed.



(i) One of the paths from A to H is A-B-D-H.
List the other two possible paths from A to H.

(ii) Find the probability that the marble passes through H or J.

(iii) Find the probability that the marble lands at N.

(iv) Two marbles fall from A, one after the other, without affecting each other.
Find the probability that they both land at P.

(c) The real numbers a, b and c have mean μ and standard deviation σ.

(i) Show that the mean of the numbers $\frac{a-\mu}{\sigma}$, $\frac{b-\mu}{\sigma}$ and $\frac{c-\mu}{\sigma}$ is 0.

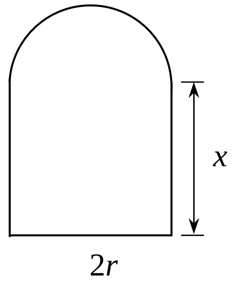
(ii) Find, with justification, the standard deviation of the numbers
$\frac{a-\mu}{\sigma}$, $\frac{b-\mu}{\sigma}$ and $\frac{c-\mu}{\sigma}$.

SECTION B

Answer one question from this section

8. **(a)** Use integration by parts to find $\int x \sin x \, dx$.

(b) A window is in the shape of a rectangle with a semicircle on top. The radius of the semicircle is r metres and the height of the rectangular part is x metres. The perimeter of the window is 20 metres.



(i) Use the perimeter to express x in terms of r and π.

(ii) Find, in terms of π, the value of r for which the area of the window is a maximum.

(c) The Maclaurin series for $\tan^{-1} x$ is $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

(i) Write down the general term of the series.

(ii) Use the Ratio Test to show that the series converges for $|x| < 1$.

(iii) Using the fact that $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$, and taking the first three terms in the Maclaurin series for $\tan^{-1} x$, find an approximation for π. Give your answer correct to five decimal places.

9. **(a)** Z is a random variable with standard normal distribution. Use the tables to find the value of z_1 for which $P(Z \geq z_1) = 0{\cdot}0778$.

(b) A die is biased in such a way that the probability of rolling a six is p. The other five numbers are all equally likely. This biased die and a fair die are rolled simultaneously. Show that the probability of rolling a total of 7 is independent of p.

(c) The mean percentage mark for candidates in the 2010 Leaving Certificate Higher Level Mathematics examination was 67·0%, with a standard deviation of 10·4%. The suggestion that candidates who appealed their results have, on average, similar results to all other candidates, is being investigated. A random sample of candidates who appealed is taken. The mean percentage mark of this sample is 69·3%.

(i) Show that if the sample size was 25, then this result *is not* significant at the 5% level.

(ii) Show that if the sample size was 100, then this result *is* significant at the 5% level.

(iii) What is the smallest sample size for which this result could be regarded as significant at the 5% level?

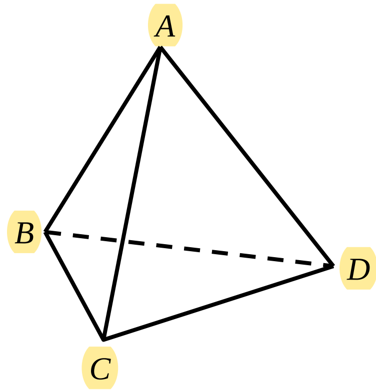
10. **(a)** A Cayley table for the group $(\{a, b, c\}, *)$ is shown.

$*$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

(i) Write down the identity element.

(ii) Write down the inverse of each element.

(b) A regular tetrahedron has twelve rotational symmetries.
These form a group G under composition, $\circ$.
The symmetries can be represented as permutations of the vertices A, B, C and D.



(i) Write down in permutation form, one element x of order 3, and describe this symmetry geometrically.

(ii) Write down in permutation form, one element y of order 2, and describe this symmetry geometrically

(iii) Show that $x \circ y \neq y \circ x$.

(iv) Let S be the set $\{e,\ x,\ y,\ x \circ y,\ y \circ x,\ x \circ x\}$, where e is the identity transformation.
Show that S is **not** closed under $\circ$.

(v) Let H be a subgroup of G. Let $x \in H$ and $y \in H$. Show that $H = G$.

11. **(a)** An ellipse has centre $(0, 0)$ and eccentricity $\frac{1}{2}$. One focus is at (2, 0).
Find the equation of the ellipse.

(b) **(i)** $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points such that $x_1 < x_2$.
If the slope of PQ is $\tan\theta$, and the length of $[PQ]$ is d, express $(x_2 - x_1)$ and $(y_2 - y_1)$ in terms of d and θ.

(ii) Let f be the transformation $(x, y) \to (x', y')$, where $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

Show that $\dfrac{|f(P)f(Q)|}{|PQ|} = \sqrt{(2\cos\theta + 5\sin\theta)^2 + (3\cos\theta + 4\sin\theta)^2}$.

(iii) Deduce that the ratio of lengths on parallel lines is invariant under f.

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