SMC V [2014-2015]: Problem Set 9 - Solutions

1

$$f(x) = 2^x$$

$$f(x+3) = 2^{x+3} \quad \text{and} \quad f(x+1) = 2^{x+1}$$

$$f(x+3) - f(x+1) = k\, f(x)$$

$$2^{x+3} - 2^{x+1} = k\, 2^x$$
$$2^x \cdot 2^3 - 2^x \cdot 2^1 = k\, 2^x$$
$$2^x (2^3 - 2^1) = k\, 2^x$$
$$\Rightarrow 8 - 2 = k$$
$$\Rightarrow k = 6$$

2

$$\log_2 x - \log_2 2 = \log_2 (1-y)$$

$$\log_2 \left(\frac{x}{2}\right) = \log_2 (1-y)$$

$$x = 2 - 2y \quad \ldots\ldots (1)$$

$$\log_2 x + \log_2 (x+2y) = 3$$
$$x(x+2y) = 2^3$$
$$x^2 + 2xy = 8 \quad \ldots\ldots (2)$$

$$x^2 + 2xy = 8$$
$$(2-2y)^2 + 2(2-2y)y = 8$$
$$4 - 8y + 4y^2 + 4y - 4y^2 = 8$$
$$-4y = 4$$
$$y = -1$$

$$x = 2 - 2y = 2 - 2(-1) = 4$$

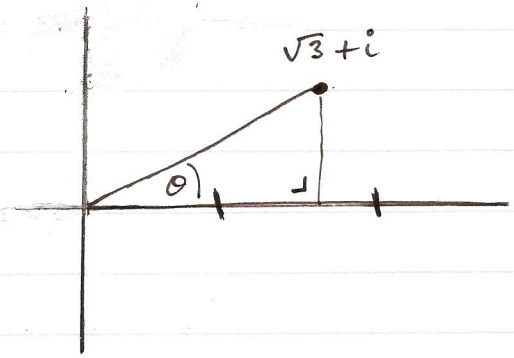
Solution: $x = 4$, $y = -1$

3//

$\sqrt{3}+i$

$r = \sqrt{(\sqrt{3})^2 + (1)^2}$

$r = 2$



$\text{Tan}\,\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta = \text{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta = 30^\circ = \frac{\pi}{6}$

$\therefore \sqrt{3}+i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$z^6 + 64$

$= \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^6 + 64$

$= 2^6\left(\cos\frac{6\pi}{6} + i\sin\frac{6\pi}{6}\right) + 64$

$= 64(\cos\pi + i\sin\pi) + 64$

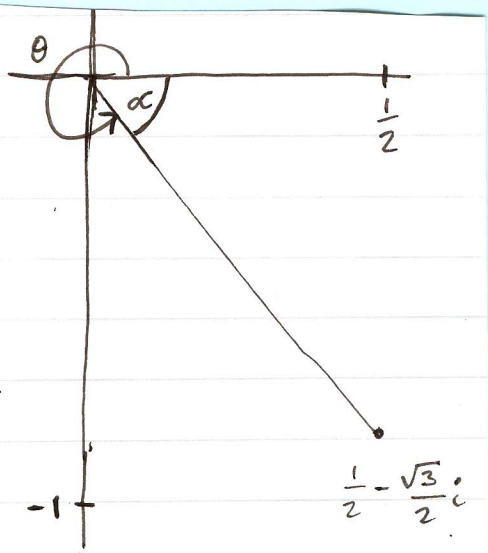
$= 64(-1 + i(0)) + 64$

$= -64 + 64$

$= 0$

$\Rightarrow \sqrt{3}+i$ is a root of $z^6 + 64 = 0$

4// $\dfrac{1-\sqrt{3}\,i}{2} = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$



$\text{Tan}\,\alpha = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

$\alpha = \text{Tan}^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$

$\Rightarrow \theta = 2\pi - \dfrac{\pi}{3} = \dfrac{5\pi}{3}$

$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$

$\therefore \dfrac{1-\sqrt{3}\,i}{2} = 1\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$\left(\dfrac{1-\sqrt{3}\,i}{2}\right)^{13} = \left[1\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\right]^{13}$

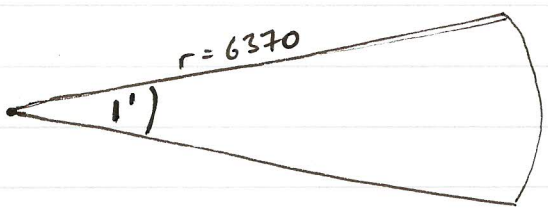
$= 1^{13}\left(\cos\frac{65\pi}{3} + i\sin\frac{65\pi}{3}\right)$

$= \cos 21\tfrac{2}{3}\pi + i\sin 21\tfrac{2}{3}$

$= \cos 1\tfrac{2}{3} + i\sin 1\tfrac{2}{3}$

$= \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

5// (a) $1' = \dfrac{1}{60}$ degree $= \dfrac{1}{60}\left(\dfrac{\pi}{180}\right)$ radians



$l = r\theta = 6370\left(\dfrac{\pi}{60(180)}\right) = 1.853$ km

(b) $v = \dfrac{d}{t} \Rightarrow t = \dfrac{d}{v} = \dfrac{2130 \div 1.853}{480} = 2.4$ hrs